CONVERGENCE OF A MASS CONSERVING ALLEN-CAHN EQUATION WHOSE LAGRANGE MULTIPLIER IS NONLOCAL *AND* LOCAL

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ABSTRACT. We consider the mass conserving Allen-Cahn equation proposed in [8]: the Lagrange multiplier which ensures the conservation of the mass contains not only nonlocal but also local effects (in contrast with [14]). As a parameter related to the thickness of a diffuse internal layer tends to zero, we perform formal asymptotic expansions of the solution. Then, equipped with this approximate solution, we rigorously prove the convergence to the volume preserving mean curvature flow, under the assumption that a classical solution of the latter exists. This requires a precise analysis of the error between the actual and the approximate Lagrange multipliers.

1. INTRODUCTION

Setting of the problem. In this paper, we consider $u_\varepsilon = u_\varepsilon(x,t)$ the solution of an Allen-Cahn equation with conservation of the mass proposed in [8], namely

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}\left(f(u_\varepsilon) - \frac{\int_\Omega f(u_\varepsilon)}{\int_\Omega \sqrt{4W(u_\varepsilon)}}\sqrt{4W(u_\varepsilon)}\right) \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

supplemented with the homogeneous Neumann boundary conditions

$$\frac{\partial u_\varepsilon}{\partial \nu}(x,t) = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.2}$$

and the initial conditions

$$u_\varepsilon(x,0) = g_\varepsilon(x) \quad \text{in } \Omega. \tag{1.3}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and ν is the Euclidian unit normal vector exterior to $\partial\Omega$. The small parameter $\varepsilon > 0$ is related to the thickness of a diffuse interfacial layer. The term

$$-\frac{\int_\Omega f(u_\varepsilon(x,t))\,dx}{\int_\Omega \sqrt{4W(u_\varepsilon(x,t))}\,dx}\sqrt{4W(u_\varepsilon(x,t))} \tag{1.4}$$

can be understood as a Lagrange multiplier for the mass constraint

$$\frac{d}{dt}\int_\Omega u_\varepsilon(x,t)\,dx = 0. \tag{1.5}$$

Let us notice that (1.4) combines nonlocal *and* local effects (see below).

The nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = \pm 1$. More precisely we assume that f is C^2 and has exactly three zeros $-1 < 0 < +1$ such that

$$f'(\pm 1) < 0, \quad f'(0) > 0 \quad \text{(bistable nonlinearity)}, \tag{1.6}$$

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and

$$f(-u) = -f(u) \quad \text{(odd nonlinearity)}. \tag{1.7}$$

The condition (1.6) implies that the potential $W(u)$ attains its local minima at $u = \pm 1$, and (1.7) implies that $W(-1) = W(+1)$, so that the two stable zeros of f, namely ± 1, have "balanced" stability. For the sake of clarity, in the computations we restrict ourselves to the case where

$$f(u) = u(1-u^2), \quad W(u) = \frac{1}{4}(1-u^2)^2. \tag{1.8}$$

This will simplify the presentation of the asymptotic expansions and is enough to capture all the features of the problem. For a more general odd and bistable nonlinearity, one would only has to make additional expansions of $f(u)$ in Section 4 and Section 6.

Remark 1.1. A more general assumption than (1.7) ensuring balanced stability is $\int_{-1}^{+1} f = 0$. In this case, this is not clear whether or not our result applies. For instance an additional term will appear in (4.35) and so in (4.46), so that $h_1 \equiv 0$ in (5.2) may fail. Since this last property is the main reason for introducing equation (1.1) (see below), we did not go further into the proof for this more general case.

The initial data g_ε are *well-prepared* in the sense that they already have sharp transition layers whose profile depends on ε. The precise assumptions on g_ε will appear in (2.10). For the moment, it is enough to note that $-1 \leq g_\varepsilon \leq 1$ and that, for a subsequence $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} g_\varepsilon = \begin{cases} -1 & \text{a.e. in the region enclosed by } \Gamma_0 \\ +1 & \text{a.e. in the region enclosed between } \partial\Omega \text{ and } \Gamma_0, \end{cases} \tag{1.9}$$

where $\Gamma_0 \subset\subset \Omega$ is a given smooth bounded hypersurface without boundary.

Our goal is to investigate the behavior of the solution u_ε of (1.1), (1.2), (1.3), as $\varepsilon \to 0$.

Related works and comments. It is long known that, even for not well-prepared initial data, the sharp interface limit of the Allen-Cahn equation $\partial_t u_\varepsilon = \Delta u_\varepsilon + \varepsilon^{-2} f(u_\varepsilon)$ moves by its mean curvature. As long as the classical motion by mean curvature exists, it was proved in [12] and an optimal estimate of the thickness of the transition layers was provided in [2]. Let us also mention that, recently, the first term of the actual profile of the layers was identified [3]. If the mean curvature flow develops singularities in finite time, then a generalized motion can be defined via level-set methods and viscosity solutions, [18] and [15]. In this framework, the convergence of the Allen-Cahn equation to generalized motion by mean curvature was proved by Evans, Soner and Souganidis [17] and a convergence rate was obtained in [1].

The above results rely on the construction of efficient sub- and super-solutions. Nevertheless, when comparison principle does not hold, a different method exists for well-prepared initial data. It was used e.g. by Mottoni and Schatzman [24] for the Allen-Cahn equation (without using the comparison principle!); Alikakos, Bates and Chen [4] for the convergence of the Cahn-Hilliard equation

$$\partial_t u_\varepsilon + \Delta\left(\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} f(u_\varepsilon)\right) = 0, \tag{1.10}$$

to the Hele-Shaw problem; Caginalp and Chen [10] for the phase field system... The idea is to first construct a solution $u_{\varepsilon,k}$ of an approximate problem thanks to matched asymptotic expansions. Next, using the lower bound of a linearized operator around such a constructed solution, an estimate of the error $\|u_\varepsilon - u_{\varepsilon,k}\|_{L^p}$ is obtained for some $p \geq 2$.

Using these technics, Chen, Hilhorst and Logak [14] considered the Allen-Cahn equation with conservation of the mass

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}\left(f(u_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega f(u_\varepsilon)\right), \tag{1.11}$$

proposed by [25] as a model for phase separation in binary mixture. They proved its convergence to the volume preserving mean curvature flow

$$V_n = -\kappa + \frac{1}{|\Gamma_t|}\int_{\Gamma_t} \kappa \, dH^{n-1} \quad \text{on } \Gamma_t. \tag{1.12}$$

Here V_n denotes the velocity of each point of Γ_t in the normal exterior direction and κ the sum of the principal curvatures, i.e. $N-1$ times the mean curvature. For related results, we also refer the reader to the works [9] (radial case, energy estimates) and [22] (case of a system).

In a recent work, Brassel and Bretin [8] proposed the mass conserving Allen-Cahn equation (1.1) as an approximation for mean curvature flow with conservation of the volume (1.12). According to their formal approach and numerical computations, it seems that "(1.1) has better volume preservation properties than (1.11)". In other words, for the approximation of mean curvature with volume constraint, they numerically observe an $\mathcal{O}(\varepsilon^2)$ error for the conservation of the volume using (1.1), whereas an $\mathcal{O}(\varepsilon)$ error is observed when using (1.11). This is clearly related to the cancellation of the ε-terms in the forthcoming expansions, see (4.17), (5.2) and Remark 1.1. Let us notice that, as far as the local Allen-Cahn equation is concerned, such an improvement of the accuracy of phase field solutions, thanks to an adequate perturbation term, was already performed in [20] or in [11].

In the present paper we prove the convergence of (1.1) to (1.12). Observe that in (1.11) the conservation of the mass (1.5) is ensured by the Lagrange multiplier $-\frac{1}{|\Omega|}\int_\Omega f(u_\varepsilon)$ which is nonlocal, whereas in the considered equation (1.1) the Lagrange multiplier (1.4) combines nonlocal and local effects. On the one hand, this will make the outer expansion completely independent of the inner one, and will cancel the ε order terms of all expansions (see Section 4). On the other hand, this makes the proof of Theorem 2.3 much more delicate since further accurate estimates are needed (see subsection 6.1). In other words, in the study [14] of (1.11), it turns out that the nonlocal Lagrange multipliers "disappear" while estimating the error estimate $u_\varepsilon - u_{\varepsilon,k}$. This will not happen in our context and our key point will be the following. Roughly speaking, our estimates of subsection 6.1 will make appear an integral of the error *on* the limit hypersurface which must be compared with the L^2 norm of the error. If the former is small compared with the latter then the Gronwall's lemma is enough. If, as expected, the error concentrates so that the former becomes large compared with the latter, then the situation is favorable: a "sign minus" intends at decreasing the L^2 norm of the error (see subsection 6.1 and Remark 6.2 for details).

To conclude let us mention the work of Golovaty [21], where a related equation with a nonlocal/local Lagrange multiplier is considered. The convergence to a weak (via viscosity solutions) volume preserving motion by mean curvature is proved via energy estimates. The author takes advantage of the fact that, under the mass constraint, the equation he considers is the gradient flow of the same energy functional as its local counterpart, namely $\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}W(u)\right)$. The equation (1.1) we consider here does not have such a property. We therefore use different methods which, moreover, allow to capture a fine error estimate between the actual solution and the constructed approximate solution.

2. Statement of the results

The flow (1.12)**.** Let us first recall a few interesting features of the averaged mean curvature flow (1.12). It is volume preserving, area shrinking and every Euclidian sphere is an equilibrium. The local in time well posedness in a classical framework is well understood (see Lemma 2.1 for a statement which is sufficient for our purpose). It is also known that local classical solutions with convex initial data turn out to be global. Additionally, there exist non-convex hypersurfaces (close to spheres) whose flow is global. For more details on the averaged mean curvature flow (1.12), we refer the reader to [19], [23], [16] and the references therein.

Lemma 2.1 (Volume preserving mean curvature flow)**.** *Let $\Omega_0 \subset\subset \Omega$ be a subdomain such that $\Gamma_0 := \partial\Omega_0$ is a smooth hypersurface without boundary. Then there is $T^{max} \in (0, \infty]$ such that the averaged mean curvature flow* (1.12)*, starting from Γ_0, has a unique smooth solution $\cup_{0\le t<T^{max}}(\Gamma_t \times \{t\})$ such that $\Gamma_t \subset\subset \Omega$, for all $t \in [0, T^{max})$.*

In the sequel, for Γ_0 as in (1.9), we fix $0 < T < T^{max}$ and work on $[0, T]$. We define

$$\Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\}),$$

and denote by Ω_t the region enclosed by Γ_t. Let us define the step function $\tilde{u} = \tilde{u}(x, t)$ by

$$\tilde{u}(x,t) := \begin{cases} -1 & \text{in } \Omega_t \\ +1 & \text{in } \Omega \setminus \overline{\Omega_t} \end{cases} \quad \text{for all } t \in [0,T], \tag{2.1}$$

which represents the sharp interface limit of u_ε as $\varepsilon \to 0$. Let d be the signed distance function to Γ defined by

$$d(x,t) = \begin{cases} -\text{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t \\ \ \ \text{dist}(x,\Gamma_t) & \text{for } x \in \Omega \setminus \overline{\Omega_t}. \end{cases} \tag{2.2}$$

Main results. Let us notice that, since $-1 \le g_\varepsilon \le 1$, it follows from the maximum principle that $-1 \le u_\varepsilon \le 1$. Also since $g_\varepsilon \not\equiv 1$ and $g_\varepsilon \not\equiv -1$, the conservation of the mass implies $u_\varepsilon \not\equiv 1$ and $u_\varepsilon \not\equiv -1$. This enables to rewrite equation (1.1) as

$$\partial_t u_\varepsilon - \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\left(f(u_\varepsilon) - \varepsilon\lambda_\varepsilon(t)(1 - {u_\varepsilon}^2)\right) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{2.3}$$

by defining

$$\varepsilon\lambda_\varepsilon(t) := \frac{\int_\Omega f(u_\varepsilon)}{\int_\Omega \sqrt{4W(u_\varepsilon)}} = \frac{\int_\Omega u_\varepsilon - {u_\varepsilon}^3}{\int_\Omega 1 - {u_\varepsilon}^2}. \tag{2.4}$$

Our first main result consists in constructing an accurate approximate solution.

Theorem 2.2 (Approximate solution)**.** *Let us fix an arbitrary integer $k > \max(N, 4)$. Then there exists $(u_{\varepsilon,k}(x,t), \lambda_{\varepsilon,k}(t))_{x\in\bar{\Omega},\, 0\le t\le T}$ such that*

$$\partial_t u_{\varepsilon,k} - \Delta u_{\varepsilon,k} - \frac{1}{\varepsilon^2}\left(f(u_{\varepsilon,k}) - \varepsilon\lambda_{\varepsilon,k}(t)(1 - {u_{\varepsilon,k}}^2)\right) = \delta_{\varepsilon,k} \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

with

$$\|\delta_{\varepsilon,k}\|_{L^\infty(\Omega\times(0,T))} = \mathcal{O}(\varepsilon^k) \quad \text{as } \varepsilon \to 0, \tag{2.6}$$

and

$$\frac{\partial u_{\varepsilon,k}}{\partial\nu}(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.7}$$

$$\frac{d}{dt}\int_\Omega u_{\varepsilon,k}(x,t)\,dx = 0 \quad \text{for all } t \in (0,T). \tag{2.8}$$

Observe that by integrating (2.5) over Ω and using (2.7) and (2.8), we see that

$$\varepsilon\lambda_{\varepsilon,k}(t) = \frac{\int_\Omega f(u_{\varepsilon,k}) + \mathcal{O}(\varepsilon^{k+2})}{\int_\Omega 1 - {u_{\varepsilon,k}}^2}. \tag{2.9}$$

Then we prove the following estimate, in the L^2 norm, on the error between the approximate solution $u_{\varepsilon,k}$ and the solution u_ε.

Theorem 2.3 (Error estimate). *Let us fix an arbitrary integer $k > \max(N,4)$. Let u_ε be the solution of* (1.1)*,* (1.2)*,* (1.3) *with the initial conditions satisfying*

$$g_\varepsilon(x) = u_{\varepsilon,k}(x,0) + \phi_\varepsilon(x) \in [-1,1], \qquad \int_\Omega \phi_\varepsilon = 0, \quad \|\phi_\varepsilon\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{k-\frac{1}{2}}). \tag{2.10}$$

Then, there is $C>0$ such that, for $\varepsilon > 0$ small enough,

$$\sup_{0\le t\le T} \|u_\varepsilon(\cdot,t) - u_{\varepsilon,k}(\cdot,t)\|_{L^2(\Omega)} \le C\varepsilon^{k-\frac{1}{2}}.$$

As it will be clear from our construction in Section 5, the approximate solution satisfies

$$\|u_{\varepsilon,k} - \tilde u\|_{L^\infty(\{(x,t):\, |d(x,t)|\ge\sqrt{\varepsilon}\})} = \mathcal{O}(\varepsilon^{k+2}), \qquad \text{as } \varepsilon \to 0,$$

with $\tilde u$ the sharp interface limit defined in (2.1) via the volume preserving mean curvature flow (1.12) starting from Γ_0. We can therefore interpret Theorem 2.3 as a result of convergence of the mass conserving Allen-Cahn equation (1.1) to the volume preserving mean curvature flow (1.12):

$$\sup_{0\le t\le T} \|u_\varepsilon(\cdot,t) - \tilde u(\cdot,t)\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{1/4}), \qquad \text{as } \varepsilon \to 0.$$

Organization of the paper. The organization of this paper is as follows. In Section 3 we present the needed tools which are by now rather classical. In Section 4, we perform formal asymptotic expansions of the solution $(u_\varepsilon(x,t), \lambda_\varepsilon(t))$. This will enable to construct the approximate solution $(u_{\varepsilon,k}(x,t), \lambda_{\varepsilon,k}(t))$, and so to prove Theorem 2.2, in Section 5. Last we prove the error estimate of Theorem 2.3 in Section 6. In particular and as mentioned before, a precise understanding of the error between the actual and the approximate Lagrange multipliers will be necessary (see subsection 6.1).

Remark 2.4. Through the paper, the notation $\psi_\varepsilon \approx \sum_{i\ge 0} \varepsilon^i \psi_i$ represents asymptotic expansion as $\varepsilon \to 0$ and means that, for all integer k, $\psi_\varepsilon = \sum_{i=0}^k \varepsilon^i\psi_i + \mathcal{O}(\varepsilon^{k+1})$.

3. Preliminaries

For the present work to be self-contained, we recall here a few properties which are classical in the works mentioned in the introduction, [25], [4], [24], [10], [11], [14], [22], and the references therein.

3.1. **Some related linearized operators.** We denote by $\theta_0(\rho) := \tanh(\frac{\rho}{\sqrt 2})$ the standing wave solution of

$$\begin{cases} \theta_0'' + f(\theta_0) = 0 \quad \text{on } \mathbb{R}, \\ \theta_0(-\infty) = -1, \quad \theta_0(0) = 0, \quad \theta_0(\infty) = 1, \end{cases}$$

which we expect to describe the transition layers of the solution u_ε observed in the stretched variable. Note that, for all $m \in \mathbb{N}$,

$$D_\rho^m[\theta_0(\rho) - (\pm 1)] = \mathcal{O}(e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty. \tag{3.1}$$

We then consider the one-dimensional underlying linearized operator around θ_0, acting on functions depending on the variable ρ by

$$\mathcal{L}u := -u_{\rho\rho} - f'(\theta_0(\rho))u. \tag{3.2}$$

Lemma 3.1 (Solvability condition and decay at infinity). *Let $A(\rho,s,t)$ be a smooth and bounded function on $\mathbb{R}\times U \times [0,T]$, with $U \subset \mathbb{R}^{N-1}$ a compact set. Then, for given $(s,t) \in U\times[0,T]$, the problem*

$$\begin{cases} \mathcal{L}\psi := -\psi_{\rho\rho} - f'(\theta_0(\rho))\psi = A(\rho,s,t) \quad \text{on } \mathbb{R}, \\ \psi(0,s,t) = 0, \quad \psi(\cdot,s,t) \in L^\infty(\mathbb{R}), \end{cases}$$

has a solution (which is then unique) if and only if

$$\int_{\mathbb{R}} A(\rho,s,t){\theta_0}'(\rho)\,d\rho = 0. \tag{3.3}$$

Under the condition (3.3)*, assume moreover that there are real constants* $A^{\pm}$ *and an integer* i *such that, for all integers* m*,* n*,* l*,*

$$D_\rho^m D_s^n D_t^l[A(\rho,s,t) - A^{\pm}] = \mathcal{O}(|\rho|^i e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{3.4}$$

uniformly in $(s,t) \in U \times [0,T]$*. Then*

$$D_\rho^m D_s^n D_t^l[\psi(\rho,s,t) + \frac{A^{\pm}}{f'(\pm 1)}] = \mathcal{O}(|\rho|^i e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{3.5}$$

uniformly in $(s,t) \in U \times [0,T]$*.*

Proof. The lemma is rather standard (see [4], [2] among others) and we only give an outline of the proof. Multiplying the equation by ${\theta_0}'$ and integrating it by parts, we easily see that the condition (3.3) is necessary. Conversely, suppose that this condition is satisfied. Then, since ${\theta_0}'$ is a bounded positive solution to the homogeneous equation $\psi_{\rho\rho} + f'(\theta_0(\rho))\psi = 0$, one can use the method of variation of constants to find the above solution ψ explicitly:

$$\psi(\rho,s,t) = {\theta_0}'(\rho)\int_0^\rho \left({\theta_0}'^{-2}(\zeta)\int_\zeta^\infty A(\xi,s,t){\theta_0}'(\xi)\,d\xi\right) d\zeta.$$

Using this expression along with the estimates (3.4) and (3.1), one then proves (3.5). □

Note also, that after the construction of the approximate solution $u_{\varepsilon,k}$, we shall need the estimate of the lower bound of the spectrum of a perturbation of the self-adjoint operator $-\Delta - \varepsilon^{-2} f'(u_{\varepsilon,k})$ proved in [13]. This will be stated in Section 6.

3.2. **Geometrical preliminaries.** The following geometrical preliminaries are borrowed from [14], to which we refer for more details and proofs.

Parametrization around Γ**.** As mentioned before, we call $\Gamma = \cup_{0\le t\le T}(\Gamma_t \times \{t\})$ the smooth solution of the volume preserving mean curvature flow (1.12), starting from Γ_0; we also denote by Ω_t the region enclosed by Γ_t. Let d be the signed distance function to Γ defined by

$$d(x,t) = \begin{cases} -\text{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t \\ \ \ \text{dist}(x,\Gamma_t) & \text{for } x \in \Omega \setminus \overline{\Omega_t}. \end{cases} \tag{3.6}$$

We remark that d is smooth in a tubular neighborhood of Γ, say in

$$\mathcal{N}_{3\delta}(\Gamma_t) := \{x \in \Omega :\ |d(x,t)| < 3\delta\},$$

for some $\delta > 0$. We choose a parametrization of Γ_t by $X_0(s,t)$, with $s \in U \subset \mathbb{R}^{N-1}$. We denote by $n(s,t)$ the unit outer normal vector on $\partial\Omega_t = \Gamma_t$. For any $0 \le t \le T$, one can then define a diffeomorphism from $(-3\delta, 3\delta) \times U$ onto the tubular neighborhood $\mathcal{N}_{3\delta}(\Gamma_t)$ by

$$X(r,s,t) = X_0(s,t) + rn(s,t) = x \in \mathcal{N}_{3\delta}(\Gamma_t),$$

whose inverse is denoted by $r = d(x,t)$, $s = S(x,t) := (S^1(x,t), \cdots, S^{N-1}(x,t))$. Then ∇d is constant along the normal lines to Γ_t, and the projection $S(x,t)$ from x on Γ_t is given by $X_0(S(x,t),t) = x - d(x,t)\nabla d(x,t)$. For $x = X_0(s,t) \in \Gamma_t$ denote by $\kappa_i(s,t)$ the principal curvatures of Γ_t at point x and by $V(s,t) := (X_0)_t(s,t).n(s,t)$ the normal velocity of Γ_t at point x. Then, one can see that

$$\kappa(s,t) := \sum_{i=1}^{N-1} \kappa_i(s,t) = \Delta d(X_0(s,t),t), \tag{3.7}$$

$$b_1(s,t) := -\sum_{i=1}^{N-1} {\kappa_i}^2(s,t) = -(\nabla d.\nabla\Delta d)(X_0(s,t),t), \tag{3.8}$$

$$V(s,t) := (X_0)_t(s,t).n(s,t) = -d_t(X(r,s,t),t). \tag{3.9}$$

In particular, $d_t(x,t)$ is independent of $r = d(x,t)$ in a small enough tubular neighborhood of Γ_t. Changing coordinates form (x,t) to (r,s,t), to any function $\phi(x,t)$ one can associate the function $\tilde\phi(r,s,t)$ by

$$\tilde\phi(r,s,t) = \phi(X_0(s,t) + rn(s,t),t) \quad \text{or} \quad \phi(x,t) = \tilde\phi(d(x,t),S(x,t),t).$$

The stretched variable. In order to describe the sharp transition layers of the solution u_ε around the limit interface, we now introduce a stretched variable. Let us consider a graph over Γ_t of the form

$$\Gamma_t^\varepsilon = \{X(r,s,t) :\, r = \varepsilon h_\varepsilon(s,t)\,,s\in U\},$$

which is expected to represent the 0 level set, at time t, of the solution u_ε. We define the stretched variable $\rho(x,t)$ as "the distance from x to Γ_t^ε in the normal direction, divided by ε", namely

$$\rho(x,t) := \frac{d(x,t) - \varepsilon h_\varepsilon(S(x,t),t)}{\varepsilon}. \tag{3.10}$$

In the sequel, we use (ρ,s,t) as independent variables for the inner expansion. The link between the old and the new variable is

$$x = \hat X(\rho,s,t) := X\left(\varepsilon(\rho + h_\varepsilon(s,t)),s,t\right) = X_0(s,t) + \varepsilon(\rho + h_\varepsilon(s,t))n(s,t).$$

Changing coordinates form (x,t) to (ρ,s,t), to any function $\psi(x,t)$ one can associate the function $\hat\psi(\rho,s,t)$ by

$$\hat\psi(\rho,s,t) = \psi(X_0(s,t) + \varepsilon(\rho + h_\varepsilon(s,t))n(s,t),t), \tag{3.11}$$

or $\psi(x,t) = \hat\psi(\frac{d(x,t)-\varepsilon h_\varepsilon(S(x,t),t)}{\varepsilon}, S(x,t),t)$. A computation then yields

$$\begin{aligned}\varepsilon^2(\partial_t\psi - \Delta\psi) = &-\hat\psi_{\rho\rho} - \varepsilon(V+\Delta d)\hat\psi_\rho \\ &+\varepsilon^2[\partial_t^\Gamma\hat\psi - \Delta^\Gamma\hat\psi - (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)\hat\psi_\rho] \\ &+\varepsilon^2[2\nabla^\Gamma h_\varepsilon.\nabla^\Gamma\hat\psi_\rho - |\nabla^\Gamma h_\varepsilon|^2\hat\psi_{\rho\rho}].\end{aligned} \tag{3.12}$$

where

$$\partial_t^\Gamma := \partial_t + \sum_{i=1}^{N-1} S_t^i\partial_{s^i},\ \nabla^\Gamma := \sum_{i=1}^{N-1}\nabla S^i\partial_{s^i},\ \Delta^\Gamma := \sum_{i=1}^{N-1}\Delta S^i\partial_{s^i} + \sum_{i,j=1}^{N-1}\nabla S^i.\nabla S^j\partial_{s^is^j}.$$

Here Δd is evaluated at $(x,t) = (X_0(s,t) + \varepsilon(\rho + h_\varepsilon(s,t))n(s,t),t)$, so that (3.7) and (3.8) imply

$$\begin{aligned}\Delta d &= \Delta d(X_0(s,t) + \varepsilon(\rho + h_\varepsilon(s,t))n(s,t),t) \\ &\approx \kappa(s,t) - \varepsilon(\rho + h_\varepsilon(s,t))b_1(s,t) - \sum_{i\geq 2}\varepsilon^i(\rho + h_\varepsilon(s,t))^i b_i(s,t),\end{aligned} \tag{3.13}$$

where $b_i(s,t)$ ($i \geq 2$) are some given functions only depending on Γ_t.

Last, define

$$\varepsilon J^\varepsilon(\rho,s,t) := \partial\hat X(\rho,s,t)/\partial(\rho,s)$$

the Jacobian of the transformation $\hat X$ so that, in particular, $dx = \varepsilon J^\varepsilon(\rho,s,t)\,dsd\rho$. Then, for all $\rho\in\mathbb{R}$, $s\in U$ and $0\leq t\leq T$, we have

$$J^\varepsilon(\rho,s,t) = \prod_{i=1}^{N-1}[1 + \varepsilon(\rho + h^\varepsilon(s,t))\kappa_i(s,t)]. \tag{3.14}$$

4. Formal asymptotic expansions

In this section, we perform formal expansions for the solution $u_\varepsilon(x,t)$ of (2.3). We start by the *outer expansion* to represent the solution "far from the limit interface", then make the *inner expansion* to describe the sharp transition layers. Last, the expansion of the nonlocal term $\lambda_\varepsilon(t)$ is performed. In the meanwhile we shall also discover the expansion of the correction term $h_\varepsilon(s,t)$ defined in (3.10).

We assume that the solution $u_\varepsilon(x,t)$ is of the form

$$u_\varepsilon(x,t) \approx u_\varepsilon^\pm(t) := \pm 1 + \varepsilon u_1^\pm(t) + \varepsilon^2 u_2^\pm(t) + \cdots \quad \text{(outer expansion)}, \tag{4.1}$$

for $x \in \Omega_t$ (corresponding to $u_\varepsilon^-(t)$), $x \in \Omega \setminus \overline{\Omega_t}$ (corresponding to $u_\varepsilon^+(t)$), and away from the interface Γ_t, say in the region where $|d(x,t)| \geq \sqrt{\varepsilon}$ as we expect the width of the transition layers to be $\mathcal{O}(\varepsilon)$. Near the interface Γ_t, i.e. in the region where $|d(x,t)| \leq \sqrt{\varepsilon}$, we assume that the function $\hat{u_\varepsilon}(\rho,s,t)$ — associated with $u_\varepsilon(x,t)$ via the change of variables (3.11)— is written as

$$\hat{u_\varepsilon}(\rho,s,t) \approx u_0(\rho,s,t) + \varepsilon u_1(\rho,s,t) + \varepsilon^2 u_2(\rho,s,t) + \cdots \quad \text{(inner expansion)}. \tag{4.2}$$

We also require the matching conditions between outer and inner expansions, that is, for all $i \in \mathbb{N}$,

$$u_i(\pm\infty,s,t) = u_i^\pm(t) \quad \text{(matching conditions)}, \tag{4.3}$$

for all $(s,t) \in U \times [0,T]$. As we expect the set $\rho = 0$ to be the 0 level set of the solution (see subsection 3.2) we impose, for all $i \in \mathbb{N}$,

$$u_i(0,s,t) = 0 \quad \text{(normalization conditions)}, \tag{4.4}$$

for all $(s,t) \in U \times [0,T]$.

As far as the nonlocal term $\lambda_\varepsilon(t)$ is concerned we assume the expansion

$$\lambda_\varepsilon(t) \approx \lambda_0(t) + \varepsilon\lambda_1(t) + \varepsilon^2\lambda_2(t) + \cdots \quad \text{(nonlocal term)}. \tag{4.5}$$

Last, the distance correcting term $h_\varepsilon(s,t)$ is assumed to be described by

$$\varepsilon h_\varepsilon(s,t) \approx \varepsilon h_1(s,t) + \varepsilon^2 h_2(s,t) + \cdots \quad \text{(distance correction term)}, \tag{4.6}$$

for all $(s,t) \in U \times [0,T]$.

In the following, by the (complete) expansion at order 1 we mean

$$\{d(x,t), \lambda_0(t), u_1(\rho,s,t), u_1^\pm(t)\} \quad \text{(expansion at order 1)},$$

and by the (complete) expansion at order $i \geq 2$ we mean

$$\{h_{i-1}(s,t), \lambda_{i-1}(t), u_i(\rho,s,t), u_i^\pm(t)\} \quad \text{(expansion at order } i \geq 2). \tag{4.7}$$

Let us also recall that we have chosen

$$f(u) = u(1-u^2), \quad W(u) = \frac{1}{4}(1-u^2)^2.$$

4.1. **Outer expansion.** By plugging the outer expansion (4.1) and the expansion (4.5) into the nonlocal partial differential equation (2.3), we get

$$\varepsilon^2(u_\varepsilon^\pm)'(t) = u_\varepsilon^\pm(t) - (u_\varepsilon^\pm(t))^3 - \varepsilon\lambda_\varepsilon(t)(1-(u_\varepsilon^\pm(t))^2). \tag{4.8}$$

Since $u_\varepsilon^\pm(t) \approx \sum_{i\geq 0} \varepsilon^i u_i^\pm(t)$, where $u_0^\pm(t) = \pm 1$, an elementary computation yields

$$-\varepsilon\lambda_\varepsilon(t)(1-(u_\varepsilon^\pm(t))^2) \approx \sum_{i\geq 1}\left(\sum_{p+q=i\,,q\neq 0} \lambda_p(t) \sum_{k+l=q} u_k^\pm(t)u_l^\pm(t)\right)\varepsilon^{i+1},$$

and

$$(u_\varepsilon^\pm(t))^3 \approx \sum_{i\geq 0}\left(\sum_{p+q=i} u_p^\pm(t) \sum_{k+l=q} u_k^\pm(t)u_l^\pm(t)\right)\varepsilon^i.$$

Hence, collecting the ε terms in (4.8), we discover $0 = u_1^\pm(t) - 3u_1^\pm(t)(u_0^\pm(t))^2$ so that $u_1^\pm(t) \equiv 0$. Next, an induction easily shows that

$$u_i^\pm(t) \equiv 0 \quad \text{for all } i \geq 1.$$

Therefore the outer expansion is already completely known and is trivial:

$$u_\varepsilon^\pm(t) \equiv \pm 1. \tag{4.9}$$

In other words, thanks to the adequate form of the Lagrange multiplier, the outer expansion is independent of the expansion of the nonlocal term. This is in contrast with the equation considered in [14].

4.2. **Inner expansion.** It follows from (3.12) that, in the new variables, equation (2.3) is recast as

$$\begin{aligned}\hat{u_\varepsilon}_{\rho\rho} + \hat{u_\varepsilon} - (\hat{u_\varepsilon})^3 = \varepsilon\lambda_\varepsilon(t)(1-(\hat{u_\varepsilon})^2) - \varepsilon(V + \Delta d)\hat{u_\varepsilon}_\rho \\ +\varepsilon^2[\partial_t^\Gamma \hat{u_\varepsilon} - \Delta^\Gamma \hat{u_\varepsilon} - (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)\hat{u_\varepsilon}_\rho] \\ +\varepsilon^2[2\nabla^\Gamma h_\varepsilon . \nabla^\Gamma \hat{u_\varepsilon}_\rho - |\nabla^\Gamma h_\varepsilon|^2 \hat{u_\varepsilon}_{\rho\rho}].\end{aligned} \tag{4.10}$$

The ε^0 terms. By collecting the ε^0 terms above and using the normalization and matching conditions (4.3), (4.4) we discover that $u_0(\rho, s, t) = \theta_0(\rho)$, with θ_0 the standing wave solution of

$$\begin{cases} \theta_0'' + f(\theta_0) = 0 \quad \text{on } \mathbb{R}, \\ \theta_0(-\infty) = -1, \quad \theta_0(0) = 0, \quad \theta_0(\infty) = 1. \end{cases} \tag{4.11}$$

Formally, this solution represents the first approximation of the profile of the transition layers around the interface observed in the stretched coordinates. Note that since $f(u) = u - u^3$, one can even compute $\theta_0(\rho) = \tanh(\frac{\rho}{\sqrt{2}})$.

The ε^1 terms. Next, since $\hat{u_\varepsilon}(\rho, s, t) \approx \sum_{i\geq 0} u_i(\rho, s, t)\varepsilon^i$, where $u_0(\rho, s, t) = \theta_0(\rho)$, an elementary computation yields

$$\varepsilon\lambda_\varepsilon(t)\left(1 - (\hat{u_\varepsilon})^2(\rho, s, t)\right) \approx -\sum_{i\geq 0}\left(\sum_{p+q=i}\lambda_p(t)\beta_q(\rho, s, t)\right)\varepsilon^{i+1}, \tag{4.12}$$

where

$$\beta_q(\rho, s, t) = \begin{cases} {\theta_0}^2(\rho) - 1 & \text{if } q = 0 \\ \sum_{k+l=q} u_k(\rho, s, t)u_l(\rho, s, t) & \text{if } q \geq 1, \end{cases}$$

and also

$$(\hat{u_\varepsilon})^3(\rho, s, t) \approx \sum_{i\geq 0}\left(\sum_{p+q=i} u_p(\rho, s, t)\sum_{k+l=q} u_k(\rho, s, t)u_l(\rho, s, t)\right)\varepsilon^i. \tag{4.13}$$

Hence, plugging the expansion (3.13) of Δd into (4.10) and collecting the ε terms, we discover

$$\mathcal{L}u_1 := -u_{1\rho\rho} - f'(\theta_0(\rho))u_1 = (V + \kappa)(s, t)\theta_0'(\rho) - (1 - {\theta_0}^2(\rho))\lambda_0(t). \tag{4.14}$$

For the above equation to be solvable (see Lemma 3.1 for details) it is necessary that, for all $(s, t) \in U \times [0, T]$,

$$\int_{\mathbb{R}} \mathcal{L}u_1(\rho, s, t)\theta_0'(\rho)\, d\rho = 0,$$

which in turn yields

$$V(s, t) = -\kappa(s, t) + \sigma\lambda_0(t), \qquad \sigma := \frac{\int_{\mathbb{R}}(1 - {\theta_0}^2)\theta_0'}{\int_{\mathbb{R}} {\theta_0'}^2}. \tag{4.15}$$

As seen in subsection 3.2 the above equation can be recast as

$$d_t(x,t) = \Delta d(x,t) - \sigma\lambda_0(t) \quad \text{for } x \in \Gamma_t. \tag{4.16}$$

Now, in view of (4.11), we can write $0 = \int_{-\infty}^{z}({\theta_0}'' + f(\theta_0)){\theta_0}' = \int_{-\infty}^{z}({\theta_0}'' - W'(\theta_0)){\theta_0}'$ and find the relation $1 - {\theta_0}^2 = \sqrt{2}{\theta_0}'$, so that $\sigma = \sqrt{2}$. Plugging this and (4.15) into (4.14) we see that $\mathcal{L}u_1 = 0$. Therefore, the normalization $u_1(0,s,t) = 0$ implies

$$u_1(\rho,s,t) \equiv 0. \tag{4.17}$$

Again this is in contrast with the equation considered in [14].

The ε^i terms ($i \geq 2$). Now, taking advantage of $u_0(\rho,s,t) = \theta_0(\rho)$ and of $u_1(\rho,s,t) \equiv 0$ we identify, for $i \geq 2$, the ε^i terms in all terms appearing in (4.10). In the sequel we omit the arguments of most of the functions and, by convention, the sum $\sum_a^b$ is null if $b < a$.

Using (4.13) we see that the ε^i term in $\hat{u_\varepsilon}_{\rho\rho} + \hat{u_\varepsilon} - (\hat{u_\varepsilon})^3$ is

$$-\mathcal{L}u_i - \theta_0 \sum_{k=2}^{i-2} u_k u_{i-k} - \sum_{p=2}^{i-2} u_p \sum_{k+l=i-p} u_k u_l \qquad \text{(term 1)}. \tag{4.18}$$

In view of (4.12), the ε^i term in $\varepsilon\lambda_\varepsilon(t)(1 - (\hat{u_\varepsilon})^2)$ is

$$\lambda_{i-1}(1 - {\theta_0}^2) - \sum_{p+q=i-1, q\neq 0} \lambda_p \sum_{k+l=q} u_k u_l \qquad \text{(term 2)}. \tag{4.19}$$

In order to deal with the term $-\varepsilon(V + \Delta d)\hat{u_\varepsilon}_\rho$, we first note that (3.13) and (4.6) yield the following expansion of the Laplacian

$$\Delta d \approx \kappa - \sum_{i\geq 1} (b_1 h_i + \delta_i)\,\varepsilon^i, \tag{4.20}$$

with

$$\delta_i = \delta_i(\rho,s,t) = \sum_{k=0}^{i} c_k(s,t)\rho^k \tag{4.21}$$

a polynomial function in ρ of degree lower than i, whose coefficients $c_k(s,t)$ are themselves polynomial in $(h_1, ..., h_{i-1})$ which are part of the formal expansion at lower orders, and in $(b_1, ...b_i)$ which are given functions. Among others, we have $\delta_1(\rho,s,t) = b_1(s,t)\rho$ and $\delta_2(\rho,s,t) = b_2(s,t)(\rho + h_1(s,t))^2$. Combining $u_{\varepsilon\rho} \approx {\theta_0}' + \varepsilon^2 u_{2\rho} + \cdots$ and (4.20), we next discover that the ε^i term in $-\varepsilon(V + \Delta d)\hat{u_\varepsilon}_\rho$ is

$$\begin{aligned} &b_1 h_{i-1}{\theta_0}' + \delta_{i-1}{\theta_0}' - (V+\kappa)\, u_{(i-1)\rho} \\ &+ \textstyle\sum_{p=1}^{i-3} (b_1 h_p + \delta_p)\, u_{(i-1-p)\rho} \qquad \text{(term 3)}. \end{aligned} \tag{4.22}$$

We see that the ε^i term in $\varepsilon^2[\partial_t^\Gamma \hat{u_\varepsilon} - \Delta^\Gamma \hat{u_\varepsilon} - (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)\hat{u_\varepsilon}_\rho]$ is given by

$$(\partial_t^\Gamma - \Delta^\Gamma)u_{i-2} - (\partial_t^\Gamma - \Delta^\Gamma)h_{i-1}{\theta_0}' - \sum_{p=1}^{i-3}(\partial_t^\Gamma - \Delta^\Gamma)h_p u_{(i-1-p)\rho} \qquad \text{(term 4)}. \tag{4.23}$$

Note that

$$|\varepsilon\nabla^\Gamma h_\varepsilon|^2 \approx \varepsilon^2|\nabla^\Gamma h_1|^2 + \sum_{i\geq 3}\left(2\nabla^\Gamma h_1.\nabla^\Gamma h_{i-1} + \eta_i\right)\varepsilon^i,$$

where

$$\eta_i = \eta_i(s,t) := \sum_{p+q=i-2, p\neq 0, q\neq 0} \nabla^\Gamma h_{p+1}(s,t).\nabla^\Gamma h_{q+1}(s,t)$$

depends only on the derivatives of h_1,...,h_{i-2}. Combining this with $\hat{u_\varepsilon}_{\rho\rho} \approx {\theta_0}'' + \varepsilon^2 u_{2\rho\rho} + \cdots$, we discover that the ε^i term in $-\varepsilon^2|\nabla^\Gamma h_\varepsilon|^2 \hat{u_\varepsilon}_{\rho\rho}$ is

$$-\beta_i(\nabla^\Gamma h_1.\nabla^\Gamma h_{i-1}){\theta_0}'' - |\nabla^\Gamma h_1|^2 u_{(i-2)\rho\rho} - \sum_{k=0}^{i-3} \alpha_k u_{k\rho\rho} \qquad \text{(term 5)}, \tag{4.24}$$

where $\alpha_k = \alpha_k(s,t)$ depends only on the derivatives of h_1,...,h_{i-2} and $\beta_2 = 0$, $\beta_i = 2$ if $i \geq 3$.

Last, since $\nabla^\Gamma \hat{u_\varepsilon}_\rho \approx \varepsilon^2 \nabla^\Gamma u_{2\rho} + \cdots$, we see that the ε^i term in $\varepsilon^2[2\nabla^\Gamma h_\varepsilon.\nabla^\Gamma \hat{u_\varepsilon}_\rho]$ is

$$2\sum_{k=2}^{i-2} \nabla^\Gamma h_{i-1-k}.\nabla^\Gamma u_{k\rho} \qquad \text{(term 6)}. \tag{4.25}$$

Hence, in view of the six terms appearing in (4.18), (4.19), (4.22), (4.23), (4.24), (4.25), when we collect the ε^i term ($i \geq 2$) in (4.10) we face up to

$$\mathcal{L}u_i = (\mathcal{M}^\Gamma h_{i-1}){\theta_0}' - (1-{\theta_0}^2)\lambda_{i-1} + \beta_i(\nabla^\Gamma h_1.\nabla^\Gamma h_{i-1}){\theta_0}'' + |\nabla^\Gamma h_1|^2 u_{(i-2)\rho\rho} + R_{i-1} \tag{4.26}$$

where $\mathcal{M}^\Gamma$ denotes the linear operator acting on functions $h(s,t)$ by

$$\mathcal{M}^\Gamma h := \partial_t^\Gamma h - \Delta^\Gamma h - b_1 h, \tag{4.27}$$

and where $R_{i-1} = R_{i-1}(\rho,s,t)$ contains all the remaining terms. Observe that, for the solvability condition for (4.26) to provide the equation (4.33) for $h_{i-1}(s,t)$, it is important that R_{i-1} does not "contain" h_{i-1}. Therefore, we have to leave the term $|\nabla^\Gamma h_1|^2 u_{(i-2)\rho\rho}$ outside R_{i-1} for the case $i = 2$, but with a slight abuse of notation we can "insert" $|\nabla^\Gamma h_1|^2 u_{(i-2)\rho\rho}$ in R_{i-1} for $i \geq 3$. As an example, for $i = 2$ we see that

$$R_1(\rho,s,t) = -\delta_1(\rho,s,t){\theta_0}'(\rho) = -b_1(s,t)\rho{\theta_0}'(\rho), \tag{4.28}$$

so that we infer that, for all integers m, n, l,

$$D_\rho^m D_s^n D_t^l[R_1(\rho,s,t)] = \mathcal{O}(|\rho|e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{4.29}$$

uniformly in (s,t). Now, for $i \geq 3$, we isolate the "worst terms" —which are the δ_i's— in R_{i-1} and write

$$R_{i-1} = -\delta_{i-1}{\theta_0}' - \sum_{p=1}^{i-3} \delta_p u_{(i-1-p)\rho} + r_{i-1}, \tag{4.30}$$

where $r_{i-1} = r_{i-1}(\rho,s,t)$ contains all the remaining terms.

Lemma 4.1 (Decay of R_{i-1})**.** *Let $i \geq 2$. Assume that, for any $1 \leq k \leq i-1$, there holds that, for all integers m, n, l,*

$$D_\rho^m D_s^n D_t^l[u_k(\rho,s,t)] = \mathcal{O}(|\rho|^{k-1}e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{4.31}$$

uniformly in $(s,t) \in U \times [0,T]$. Then, for all integers m, n, l

$$D_\rho^m D_s^n D_t^l[R_{i-1}(\rho,s,t)] = \mathcal{O}(|\rho|^{i-1}e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{4.32}$$

uniformly in $(s,t) \in U \times [0,T]$.

Proof. Let us have a look at expression (4.30) of R_{i-1}. By a tedious but straightforward examination of the six terms (4.18), (4.19), (4.22), (4.23), (4.24), (4.25), one can write the exact expression of R_{i-1} appearing in (4.26) and, so, that of r_{i-1} appearing in (4.30). In view of this exact expression (that we do not write here) and of estimates (4.31), one can see that $r_{i-1}(\rho,s,t)$ depends only on

- $V(s,t)$, $\kappa(s,t)$, $b_1(s,t)$, ..., $b_i(s,t)$ which are bounded given functions
- $\lambda_0(t)$, ..., $\lambda_{i-2}(t)$
- $h_1(s,t)$, ..., $h_{i-2}(s,t)$ and their derivatives w.r.t. s and t

- $u_0(\rho, s, t)$ which is equal to $\theta_0(\rho)$, $u_1(\rho, s, t)$ which vanishes, ..., $u_{i-1}(\rho, s, t)$ and their derivatives w.r.t. ρ, s and t

in such a way that it is $\mathcal{O}(|\rho|^{i-2}e^{-\sqrt{2}|\rho|})$ as $\rho \to \pm\infty$. Concerning the term

$$-\delta_{i-1}(\rho, s, t){\theta_0}'(\rho) - \sum_{p=1}^{i-3} \delta_p(\rho, s, t) u_{(i-1-p)\rho},$$

the fact that it behaves like (4.32) follows from (4.31) and the fact that $\delta_p(\rho, s, t)$ grows like $|\rho|^p$, as seen in (4.21). □

Now, in virtue of Lemma 3.1, the solvability condition for equation (4.26) yields, for all (s, t),

$$(\mathcal{M}^\Gamma h_{i-1})(s, t) \int_{\mathbb{R}} {\theta_0'}^2 - \lambda_{i-1}(t) \int_{\mathbb{R}} (1 - {\theta_0}^2){\theta_0}' + \int_{\mathbb{R}} R_{i-1}(\cdot, s, t){\theta_0}' = 0. \tag{4.33}$$

Note that the term $-\beta_i(\nabla^\Gamma h_1.\nabla^\Gamma h_{i-1}){\theta_0}''$ does not appear above since $\int_{\mathbb{R}} {\theta_0}''{\theta_0}' = 0$. Note also that the term $-|\nabla^\Gamma h_1|^2 u_{(i-2)\rho\rho}$ does not appear for the same reason if $i = 2$, and because it can be "inserted" in R_{i-1} for $i \geq 3$ without altering the fact that R_{i-1} does not depend on h_{i-1} (see also the explanations after (4.27)). The above equality can be recast as

$$(\mathcal{M}^\Gamma h_{i-1})(s, t) = \sigma \lambda_{i-1}(t) - \sigma^* \int_{\mathbb{R}} R_{i-1}(\rho, s, t){\theta_0}'(\rho)\, d\rho, \tag{4.34}$$

with σ defined in (4.15) and $\sigma^* := \left(\int_{\mathbb{R}} {\theta_0'}^2\right)^{-1}$. Note that, thanks to $1 - {\theta_0}^2 = \sqrt{2}{\theta_0}'$, we have $\sigma = \sqrt{2}$ (as seen before) and also $\sigma^* = \frac{3}{4}\sqrt{2}$.

In order to construct the terms u_i for $i \geq 2$ by induction, let us first examine the case $i = 2$. From (4.28) and the fact that $\int_{\mathbb{R}} \rho {\theta_0'}^2(\rho)\, d\rho = 0$ (odd function), we see that (4.34) reduces to

$$(\mathcal{M}^\Gamma h_1)(s, t) = \sigma \lambda_1(t). \tag{4.35}$$

Assume that h_1 satisfies the above equation. Then since $u_1 \equiv 0$ trivially satisfies (4.31), Lemma 4.1 implies that $R_1(\rho, s, t)$ together with its derivatives are $\mathcal{O}(|\rho|e^{-\sqrt{2}|\rho|})$ as $\rho \to \pm\infty$. It follows from Lemma 3.1 that

$$\mathcal{L}u_2 = (\mathcal{M}^\Gamma h_1){\theta_0}' - (1 - {\theta_0}^2)\lambda_1 + |\nabla^\Gamma h_1|^2 {\theta_0}'' + R_1, \tag{4.36}$$

admits a unique solution $u_2(\rho, s, t)$ such that $u_2(0, s, t) = 0$, which additionally satisfies $D_\rho^m D_s^n D_t^l [u_2(\rho, s, t)] = \mathcal{O}(|\rho|e^{-\sqrt{2}|\rho|})$.

Now, an induction argument straightforwardly concludes the construction of the inner expansion.

Lemma 4.2 (Construction by induction)**.** *Let $i \geq 2$. Assume that, for all $1 \leq k \leq i - 1$ the term u_k is constructed such that*

$$D_\rho^m D_s^n D_t^l [u_k(\rho, s, t)] = \mathcal{O}(|\rho|^{k-1}e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{4.37}$$

uniformly in $(s, t) \in U \times [0, T]$. Assume moreover that $h_{i-1}(s, t)$ satisfies the solvability condition (4.34)*. Then one can construct $u_i(\rho, s, t)$ solution of* (4.26) *such that $u_i(0, s, t) = 0$ and*

$$D_\rho^m D_s^n D_t^l [u_i(\rho, s, t)] = \mathcal{O}(|\rho|^{i-1}e^{-\sqrt{2}|\rho|}) \quad \text{as } \rho \to \pm\infty, \tag{4.38}$$

uniformly in $(s, t) \in U \times [0, T]$.

4.3. **Expansion of the nonlocal term $\lambda_\varepsilon(t)$ and the distance correction term $h_\varepsilon(s,t)$.** By following [14, subsection 5.4] with $\sqrt{\varepsilon}$ playing the role of δ, we see that an asymptotic expansion of the conservation of the mass (1.5) yields

$$0 = \frac{d}{dt}\int_\Omega u_\varepsilon(x,t)\,dt \approx I_1 + I_2 + I_3, \tag{4.39}$$

where $I_1 = 0$, since in our case $u_\varepsilon^\pm(t) \equiv \pm 1$, and

$$I_2 := \int_{|\rho|<1/\sqrt{\varepsilon}} \partial_t^\Gamma \hat{u_\varepsilon}(\rho,s,t)\,\varepsilon J^\varepsilon(\rho,s,t)\,d\rho\,ds, \tag{4.40}$$

$$I_3 := \int_{|\rho|<1/\sqrt{\varepsilon}} (-V - \varepsilon\partial_t^\Gamma h_\varepsilon)(s,t)\,\partial_\rho \hat{u}_\varepsilon(\rho,s,t)\,J^\varepsilon(\rho,s,t)\,d\rho\,ds, \tag{4.41}$$

Combining $\partial_t^\Gamma := \partial_t + \sum_{i=1}^{N-1} S_t^i \partial_{s^i}$ with $u_0(\rho,s,t) = \theta_0(\rho)$ and $u_1(\rho,s,t) \equiv 0$, we see that

$$\partial_t^\Gamma \hat{u_\varepsilon}(\rho,s,t) \approx \sum_{i\geq 2} \varepsilon^i [\partial_t + \sum_{k=1}^{N-1} S_t^k \partial_{s^k}] u_i(\rho,s,t).$$

In view of the above inner expansion, this implies

$$\partial_t^\Gamma \hat{u_\varepsilon}(\rho,s,t) \approx \sum_{i\geq 2} \varepsilon^i \mathcal{O}\left(|\rho|^{i-1} e^{-\sqrt{2}|\rho|}\right),$$

where $\mathcal{O}\left(|\rho|^{i-1} e^{-\sqrt{2}|\rho|}\right)$ depends only on expansions at orders $\leq i-1$. By plugging this into (4.40), we get

$$I_2 \approx \sum_{i\geq 3} \varepsilon^i \gamma_{i-2},$$

where $\gamma_{i-2} = \gamma_{i-2}(t)$ depends only on expansions at orders $\leq i-2$.

We now turn to the term I_3. We expand

$$\left(-V - \varepsilon\partial_t^\Gamma h_\varepsilon\right)(s,t) \approx d_t(X_0(s,t),t) - \sum_{i\geq 1} \varepsilon^i \partial_t^\Gamma h_i(s,t),$$

and

$$\partial_\rho \hat{u_\varepsilon}(\rho,s,t) \approx {\theta_0}'(\rho) + \sum_{i\geq 2} \varepsilon^i \partial_\rho u_i(\rho,s,t).$$

Expanding the Jacobian (3.14) and using (3.7), we get

$$J^\varepsilon(\rho,s,t) \approx 1 + \Delta d(X_0(s,t),t)\,\varepsilon(\rho + h^\varepsilon(s,t)) + \sum_{i\geq 2} \varepsilon^i \mu_{i-1},$$

where $\mu_{i-1} = \mu_{i-1}(\rho,s,t)$ depends only on expansions at orders $\leq i-1$. Multiplying the three above equalities, we see that the integrand in I_3 expands as

$${\theta_0}' d_t + \varepsilon {\theta_0}' \left[-\partial_t^\Gamma h_1 + h_1 d_t \Delta d + \rho d_t \Delta d\right] + \sum_{i\geq 2} \varepsilon^i {\theta_0}'(-\partial_t^\Gamma h_i + h_i d_t \Delta d + \upsilon_{i-1}),$$

where $\upsilon_{i-1} = \upsilon_{i-1}(\rho,s,t)$ depends only on expansions at orders $\leq i-1$. We integrate this over $s \in U$ and $|\rho| < 1/\sqrt{\varepsilon}$ and, using $\int_{|\rho|<1/\sqrt{\varepsilon}} {\theta_0}' \approx \int_{\mathbb{R}} {\theta_0}'(\rho)\,d\rho = 2$ and $\int_{|\rho|<1/\sqrt{\varepsilon}} \rho{\theta_0}'(\rho)\,d\rho = 0$ (odd function), we discover

$$\begin{aligned}\frac{1}{2} I_3 \quad &\approx \quad \int_U d_t(s,t)\,ds + \varepsilon \int_U (-\partial_t^\Gamma h_1 + (d_t\Delta d)h_1)(s,t)\,ds \\ &\quad + \sum_{i\geq 2} \varepsilon^i \left[\int_U (-\partial_t^\Gamma h_i + (d_t \Delta d) h_i)(s,t)\,ds + \omega_{i-1}\right],\end{aligned}$$

where $\omega_{i-1} = \omega_{i-1}(t)$ depends only on expansions at orders $\leq i-1$. Using (4.16) to substitute d_t, (4.35) to substitute $\partial_t^\Gamma h_1$, (4.34) to substitute $\partial_t^\Gamma h_i$, we have

$$\begin{aligned}\frac{1}{2}I_3 \quad &\approx \quad \int_U (\Delta d - \sigma\lambda_0)\, ds + \varepsilon \int_U (-\Delta^\Gamma h_1 - b_1 h_1 - \sigma\lambda_1 + (d_t\Delta d)h_1)\, ds \\ &\quad + \sum_{i\geq 2} \varepsilon^i \left[\int_U (-\Delta^\Gamma h_i - b_1 h_i - \sigma\lambda_i + (d_t\Delta d)h_i)\, ds + \zeta_{i-1}\right],\end{aligned}$$

where $\zeta_{i-1} = \zeta_{i-1}(t)$ depends only on expansions at orders $\leq i-1$.

Last, using $\int_U \Delta^\Gamma h_i\, ds = 0$, we see that $I_2 + I_3 \approx 0$ reduces to

$$\sigma\lambda_0(t) = \overline{\Delta d(\cdot,t)} \tag{4.42}$$

$$\sigma\lambda_1(t) = -\overline{[b_1(\cdot,t) - d_t(\cdot,t)\Delta d(\cdot,t)]h_1(\cdot,t)} \tag{4.43}$$

$$\sigma\lambda_i(t) = -\overline{[b_1(\cdot,t) - d_t(\cdot,t)\Delta d(\cdot,t)]h_i(\cdot,t)} + \Lambda_{i-1}(t) \quad (i \geq 2), \tag{4.44}$$

where $\overline{\phi(\cdot)} := \frac{1}{|U|}\int_U \phi$ denotes the average of ϕ over Γ_t (parametrized by U), and $\Lambda_{i-1}(t)$ depends only on expansions at orders $\leq i-1$. Moreover if we plug (4.42), (4.43) and (4.44) into (4.16), (4.35) and (4.34), we have the following closed system for d, h_1,.., h_i on $U \times [0,T]$:

$$d_t = \Delta d - \overline{\Delta d(\cdot,t)} \tag{4.45}$$

$$\partial_t^\Gamma h_1 = \Delta^\Gamma h_1 + b_1 h_1 - \overline{[b_1(\cdot,t) - d_t(\cdot,t)\Delta d(\cdot,t)]h_1(\cdot,t)} \tag{4.46}$$

$$\partial_t^\Gamma h_i = \Delta^\Gamma h_i + b_1 h_i - \overline{[b_1(\cdot,t) - d_t(\cdot,t)\Delta d(\cdot,t)]h_i(\cdot,t)} + \Lambda_{i-1}(t) \quad (i \geq 2). \tag{4.47}$$

5. The approximate solution $u_{\varepsilon,k}$, $\lambda_{\varepsilon,k}$

In order to construct our desired approximate solution and prove Theorem 2.2, let us first explain how the previous section enables to determine, at any order, the outer expansion (4.1), the inner expansion (4.2), the expansion of the nonlocal term (4.5), and the expansion of the distance correction term (4.6).

First, as seen before, the outer expansion (4.1) is already completely known since $u_i^\pm(t) \equiv 0$ for all $i \geq 1$.

Recall that $\Gamma = \cup_{0\leq t\leq T}(\Gamma_t \times \{t\})$ denotes the unique smooth evolution of the volume preserving mean curvature flow (1.12) starting from $\Gamma_0 \subset\subset \Omega$, to which we associate the signed distance function $d(x,t)$. Hence, defining $\lambda_0(t)$ as in (4.42) and $u_1(\rho,s,t) \equiv 0$ as in (4.17), we are equipped with the first order expansion

$$\{d(x,t), \lambda_0(t), u_1(\rho,s,t) \equiv 0\}. \tag{5.1}$$

Next, since Γ_t is a smooth hypersurface without boundary, there is a unique smooth solution $h_1(s,t)$ to the parabolic equation (4.46). Assuming $h_1(s,0) = 0$ for $s \in U$, we see that $h_1(s,t) \equiv 0$, which combined with (4.43) yields $\lambda_1(t) \equiv 0$. Notice that these cancellations are consistent with the observation of [8] that "(1.1) has better volume preserving properties than the traditional mass conserving Allen-Cahn equation (1.11)". In Section 4, we have defined $u_2(\rho,s,t)$ as the solution of (4.36), which now reduces to $\mathcal{L}u_2 = -b_1(s,t)\rho{\theta_0}'(\rho)$. This completes the second order expansion, namely

$$\{h_1(s,t) \equiv 0, \lambda_1(t) \equiv 0, u_2(\rho,s,t)\}. \tag{5.2}$$

Now, for $i \geq 2$, let us assume that expansions $\{h_{k-1}(s,t), \lambda_{k-1}(t), u_k(\rho,s,t)\}$ are constructed for all $2 \leq k \leq i$. Therefore we can construct $\Lambda_{i-1}(t)$ appearing in (4.47). Assuming $h_i(s,0) = 0$ for $s \in U$, there is a unique smooth solution $h_i(s,t)$ to the parabolic equation (4.47). This enables to construct $\lambda_i(t)$ via (4.44). Now, $h_i(s,t)$ satisfies the solvability condition (4.34) at rank i, so that Lemma 4.2 provides $u_{i+1}(\rho,s,t)$, the solution of (4.26) at rank $i+1$ with $u_{i+1}(0,s,t) = 0$. This completes the construction of the $i+1$-th order expansion $\{h_i(s,t), \lambda_i(t), u_{i+1}(\rho,s,t)\}$.

Note also that, from the above induction argument, we also deduce the behavior (4.38) for all the $u_i(\rho, s, t)$'s.

Proof of Theorem 2.2. We are now in the position to construct the approximate solution as stated in Theorem 2.2. Let us fix an integer $k > \max(N, 4)$. We define

$$\begin{aligned} \rho_{\varepsilon,k}(x,t) &:= \frac{1}{\varepsilon}\left[d(x,t) - \sum_{i=1}^{k+2} \varepsilon^i h_i(S(x,t),t)\right] = \frac{d_{\varepsilon,k}(x,t)}{\varepsilon}, \\ u^{in}_{\varepsilon,k}(x,t) &:= \theta_0(\rho_{\varepsilon,k}(x,t)) + \sum_{i=1}^{k+3} \varepsilon^i u_i(\rho_{\varepsilon,k}(x,t), S(x,t), t), \\ u^{out}_{\varepsilon,k}(x,t) &:= \tilde{u}(x,t), \\ \lambda_{\varepsilon,k}(t) &:= \lambda_0(t) + \sum_{i=1}^{k+2} \varepsilon^i \lambda_i(t), \end{aligned}$$

where $\tilde{u}$ is the sharp interface limit defined in (2.1). We introduce a smooth cut-off function $\zeta(z) = \zeta_\varepsilon(z)$ such that

$$\begin{cases} \zeta(z) = 1 & \text{if } |z| \le \sqrt{\varepsilon}, \\ \zeta(z) = 0 & \text{if } |z| \ge 2\sqrt{\varepsilon}, \\ 0 \le z\zeta'(z) \le 4 & \text{if } \sqrt{\varepsilon} \le |z| \le 2\sqrt{\varepsilon}. \end{cases}$$

For $x \in \bar{\Omega}$ and $0 \le t \le T$, we define

$$u^*_{\varepsilon,k}(x,t) := \zeta(d(x,t)) u^{in}_{\varepsilon,k}(x,t) + [1 - \zeta(d(x,t))] u^{out}_{\varepsilon,k}(x,t).$$

If $\varepsilon > 0$ is small enough then the signed distance $d(x,t)$ is smooth in the tubular neighborhood $\mathcal{N}_{3\sqrt{\varepsilon}}(\Gamma)$, and so is $u^{in}_{\varepsilon,k}(x,t)$. This shows that $u^*_{\varepsilon,k}$ is smooth.

Plugging $(u^*_{\varepsilon,k}(x,t), \lambda_\varepsilon(t))$ into the left-hand side of (2.5), we find a error term $\delta^*_{\varepsilon,k}(x,t)$ which is such that

- $\delta^*_{\varepsilon,k}(x,t) = 0$ on $\{|d(x,t)| \ge 2\sqrt{\varepsilon}\}$ since, then, $u^*_{\varepsilon,k} = u^{out}_{\varepsilon,k} = \pm 1$,
- $\|\delta^*_{\varepsilon,k}\|_{L^\infty} = \mathcal{O}(\varepsilon^{k+2})$ on $\{|d(x,t)| \le \sqrt{\varepsilon}\}$ since, then, $u^*_{\varepsilon,k} = u^{in}_{\varepsilon,k}$ and the expansions of Section 4 were done on this purpose,
- $\|\delta^*_{\varepsilon,k}\|_{L^\infty} = \mathcal{O}(\varepsilon^{k^*})$, for any integer k^*, on $\{\sqrt{\varepsilon} \le |d(x,t)| \le 2\sqrt{\varepsilon}\}$ since, then, the decaying estimates (3.1) and (4.38) imply that $u^*_{\varepsilon,k} - u^{out}_{\varepsilon,k} = u^*_{\varepsilon,k} - \pm 1 = \mathcal{O}(e^{-\frac{\sqrt{2}}{2\sqrt{\varepsilon}}})$, valid also after any differentiation.

Hence $\|\delta^*_{\varepsilon,k}\|_{L^\infty(\Omega\times(0,T))} = \mathcal{O}(\varepsilon^{k+2})$, which is even better than (2.5). Also $u^*_{\varepsilon,k}$ clearly satisfies (2.7).

Now, to ensure the conservation of the mass of the approximate solution, we add a correcting term (which depends only on time) and define

$$u_{\varepsilon,k}(x,t) := u^*_{\varepsilon,k}(x,t) + \frac{1}{|\Omega|}\int_\Omega (u^*_{\varepsilon,k}(x,0) - u^*_{\varepsilon,k}(x,t))\,dx,$$

which then satisfies (2.8), and still (2.7). Note also that subsection 4.3 implies that the correcting term

$$\int_\Omega (u^*_{\varepsilon,k}(x,0) - u^*_{\varepsilon,k}(x,t))\,dx = -\int_\Omega \int_0^t \partial_t u^*_{\varepsilon,k}(x,\tau)\,d\tau dx$$

is $\mathcal{O}(\varepsilon^{k+2})$ together with its time derivative. Hence, when we plug $u_{\varepsilon,k} = u^*_{\varepsilon,k} + \mathcal{O}(\varepsilon^{k+2})$ into the left-hand side of (2.5), we find a error term $\delta_{\varepsilon,k}$ whose L^∞ norm is $\mathcal{O}(\varepsilon^k)$. □

6. Error estimate

We shall here prove the error estimate, namely Theorem 2.3. For ease of notation, we drop most of the subscripts ε and write u, λ, u_k, λ_k, δ_k for u_ε, λ_ε, $u_{\varepsilon,k}$, $\lambda_{\varepsilon,k}$, $\delta_{\varepsilon,k}$ respectively. By $\|\cdot\|$, $\|\cdot\|_{2+p}$ we always mean $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{L^{2+p}(\Omega)}$ respectively. In the sequel, we denote by C various positive constants which may change from places to places and are independent on $\varepsilon > 0$.

Let us define the error

$$R(x,t) := u(x,t) - u_k(x,t).$$

Clearly $\|R\|_{L^\infty} \leq 3$. It follows from the mass conservation properties (1.5), (2.8), and the initial conditions (2.10) that

$$\int_\Omega R(x,t)\,dx = 0 \quad \text{for all } 0 \leq t \leq T, \quad \|R(\cdot,0)\| = \mathcal{O}(\varepsilon^{k-\frac{1}{2}}). \tag{6.1}$$

We successively subtract the approximate equation (2.5) from equation (2.3), multiply by R and then integrate over Ω. This yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega R^2 = &-\int_\Omega |\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega f'(u_k)R^2 \\ &+ \frac{1}{\varepsilon^2}\int_\Omega (f(u) - f(u_k) - f'(u_k)R)R - \int_\Omega \delta_k R - \frac{1}{\varepsilon^2}\Lambda, \end{aligned} \tag{6.2}$$

where

$$\Lambda = \Lambda(t) := \int_\Omega [\varepsilon\lambda(1-u^2)R - \varepsilon\lambda_k(1-{u_k}^2)R]. \tag{6.3}$$

Since $(f(u) - f(u_k) - f'(u_k)R)R = -3u_kR^3 - R^4 = \mathcal{O}(R^{2+p})$, where $p := \min(\frac{4}{N}, 1)$, we have

$$\left|\frac{1}{\varepsilon^2}\int_\Omega (f(u) - f(u_k) - f'(u_k)R)R\right| \leq \frac{1}{\varepsilon^2}C\|R\|_{2+p}^{2+p} \leq \frac{1}{\varepsilon^2}C_1\|R\|^p\|\nabla R\|^2,$$

where we have used the interpolation result [14, Lemma 1]. We also have $\left|\int_\Omega \delta_k R\right| \leq \|\delta_k\|_\infty\|R\| = \mathcal{O}(\varepsilon^k)\|R\|$, so that

$$\begin{aligned} \|R\|\frac{d}{dt}\|R\| \leq &-\int_\Omega |\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega f'(u_k)R^2 \\ &+ \frac{1}{\varepsilon^2}C_1\|R\|^p\|\nabla R\|^2 + \mathcal{O}(\varepsilon^k)\|R\| - \frac{1}{\varepsilon^2}\Lambda. \end{aligned} \tag{6.4}$$

We shall estimate Λ in the following subsection. As mentioned before, this term is the main difference with the case of a strictly nonlocal Lagrange multiplier: its analogous for equation (1.11) is $(\varepsilon\lambda - \varepsilon\lambda_k)\int_\Omega R$ which vanishes, see [14].

Since $k > \max(N,4)$ we have $k - \frac{1}{2} > \frac{4}{p} = \frac{4}{\min(\frac{4}{N},1)}$, so that the second estimate in (6.1) allows to define $t_\varepsilon > 0$ by

$$t_\varepsilon := \sup\left\{t > 0, \forall\, 0 \leq \tau \leq t, \|R(\cdot,\tau)\| \leq (2C_1)^{-1/p}\varepsilon^{4/p}\right\}. \tag{6.5}$$

We need to prove that $t_\varepsilon = T$ and that the estimate $\mathcal{O}(\varepsilon^{4/p})$ is actually improved to $\mathcal{O}(\varepsilon^{k-\frac{1}{2}})$. In the sequel we work on the time interval $[0, t_\varepsilon]$.

6.1. **Error estimates between the nonlocal/local Lagrange multipliers.** It follows from (2.9) that the term Λ under consideration is recast as

$$\Lambda = \frac{A}{B}E - \frac{A_k}{B_k}E_k + \frac{\mathcal{O}(\varepsilon^{k+2})}{B_k}E_k, \tag{6.6}$$

where

$$A_k = A_k(t) := \int_\Omega f(u_k),\; B_k = B_k(t) := \int_\Omega 1 - {u_k}^2,\; E_k = E_k(t) := \int_\Omega (1 - {u_k}^2)R,$$

and A, B, E the same quantities with u in place of u_k.

Lemma 6.1 (Some expansions)**.** *We have, as $\varepsilon \to 0$,*

$$A_k = \varepsilon^2\alpha + \mathcal{O}(\varepsilon^3), \quad B_k = \varepsilon\beta + \mathcal{O}(\varepsilon^2),$$

where

$$\alpha = \alpha(t) := \int_U \sum_{i=1}^{N-1} \kappa_i(s,t)\,ds \int_{\mathbb{R}} \rho f(\theta_0(\rho))\,d\rho, \quad \beta := 2\sqrt{2}|U|,$$

and

$$E_k = \mathcal{O}(\sqrt{\varepsilon}\|R\|).$$

Proof. We have seen in Section 5 that $u_k = u_k^* + \mathcal{O}(\varepsilon^{k+2})$ so it is enough to deal with A_k^*, B_k^* and E_k^*. The lemma is then rather clear from the expansions of Section 4. We have

$$\begin{aligned} A_k^* &= \int_{|d(x,t)|\le 2\sqrt{\varepsilon}} f(u_k^*)(x,t)\,dx = \int_{|d(x,t)|\le\sqrt{\varepsilon}} f(u_k^*)(x,t)\,dx + \mathcal{O}(e^{-\frac{\sqrt{2}}{2\sqrt{\varepsilon}}}) \\ &= \int_U \int_{|\rho|\le 1/\sqrt{\varepsilon}} f(\theta_0(\rho) + O(\varepsilon^2))\varepsilon J^\varepsilon(\rho,s,t)\,dsd\rho + \mathcal{O}(e^{-\frac{\sqrt{2}}{2\sqrt{\varepsilon}}}). \end{aligned}$$

Using $J^\varepsilon(\rho,s,t) = 1 + \varepsilon\rho\sum_{i=1}^{N-1}\kappa_i(s,t) + \mathcal{O}(\varepsilon^2)$ and $\int_{|\rho|\le 1/\sqrt{\varepsilon}} f(\theta_0(\rho))\,d\rho = 0$ (odd function), one obtains the estimate for A_k^*. The estimate for B_k^* follows the same lines and is omitted. Last, the Hölder inequality yields $|E_k| \le (\int_\Omega (1-{u_k}^2)^2)^{1/2}\|R\| = \mathcal{O}(\sqrt{\varepsilon}\|R\|)$ since, again, $dx = \varepsilon J^\varepsilon(\rho,s,t)\,ds\,d\rho$. □

As a first consequence of the above lemma, it follows from (6.6) that

$$\Lambda = \frac{A}{B}E - \frac{A_k}{B_k}E_k + \mathcal{O}(\varepsilon^{k+\frac{3}{2}})\|R\|. \tag{6.7}$$

Next, in view of the above lemma, $u = u_k + R$ and $\|R\| = \mathcal{O}(\varepsilon^{4/p})$, we can thus perform the following expansions

$$\begin{aligned} A &= A_k + \int_\Omega (1 - 3{u_k}^2)R - 3\int_\Omega u_k R^2 - \int_\Omega R^3 \\ &= A_k + 3E_k - 3\int_\Omega u_k R^2 + \mathcal{O}(\|R\|_{2+p}^{2+p}), \end{aligned}$$

since $\int_\Omega R = 0$,

$$\begin{aligned} B^{-1} &= {B_k}^{-1}\left(1 - \frac{2\int_\Omega u_k R}{B_k} - \frac{\int_\Omega R^2}{B_k}\right)^{-1} \\ &= {B_k}^{-1}\left(1 + \frac{2\int_\Omega u_k R}{B_k} + \frac{\int_\Omega R^2}{B_k} + \left(\frac{2\int_\Omega u_k R}{B_k}\right)^2 + \mathcal{O}\left(\frac{\|R\|^3}{\varepsilon^3}\right)\right), \end{aligned}$$

and

$$E = E_k - 2\int_\Omega u_k R^2 + \mathcal{O}(\|R\|_{2+p}^{2+p}).$$

It follows that, using $E_k = \mathcal{O}(\sqrt{\varepsilon}\|R\|)$ and $A_k = \mathcal{O}(\varepsilon^2)$ (see Lemma 6.1),

$$\begin{aligned}AE =& A_kE_k - 2A_k\int_\Omega u_kR^2 + \mathcal{O}(\varepsilon^2\|R\|_{2+p}^{2+p}) + 3{E_k}^2 - 6E_k\int_\Omega u_kR^2\\ &+ \mathcal{O}(\sqrt{\varepsilon}\|R\|\,\|R\|_{2+p}^{2+p}) - 3E_k\int_\Omega u_kR^2 + \mathcal{O}(\|R\|^4) + \mathcal{O}(\|R\|^2\,\|R\|_{2+p}^{2+p})\\ &+ \mathcal{O}(\|R\|_{2+p}^{2+p}\sqrt{\varepsilon}\|R\|) + \mathcal{O}(\|R\|_{2+p}^{2+p}\,\|R\|^2) + \mathcal{O}(\|R\|_{2+p}^{4+2p})\\ =& A_kE_k + 3{E_k}^2 - 9E_k\int_\Omega u_kR^2 - 2A_k\int_\Omega u_kR^2 + \mathcal{O}(\varepsilon^2\|R\|_{2+p}^{2+p}) + \mathcal{O}(\|R\|^3),\end{aligned}$$

since $\|R\|_{2+p}^{2+p} = \mathcal{O}(\|R\|^2)$. Now, using the above expressions, we aim at expanding $\frac{A}{B}E - \frac{A_k}{B_k}E_k$. For the convenience of the reader let us explain how to handle two of the terms appearing in the computations: the estimates in Lemma 6.1 yield, as $\varepsilon \to 0$,

$$\frac{2\int_\Omega u_kR}{B_k}\,3{E_k}^2 = \mathcal{O}\left(\frac{\|R\|}{\varepsilon}\right)\mathcal{O}(\varepsilon\|R\|^2) = \mathcal{O}(\|R\|^3),$$

and

$$\frac{2\int_\Omega u_kR}{B_k}\,(-9)E_k\int_\Omega u_kR^2 = \mathcal{O}\left(\frac{\|R\|}{\varepsilon}\right)\mathcal{O}(\sqrt{\varepsilon}\|R\|\,\|R\|^2) = \mathcal{O}\left(\frac{\|R\|^4}{\sqrt{\varepsilon}}\right) = \mathcal{O}(\|R\|^3),$$

the last estimate following from the definition of t_ε in (6.5). Using similar arguments to treat other terms, we obtain

$$\begin{aligned}\frac{A}{B}E - \frac{A_k}{B_k}E_k =& {B_k}^{-1}\Big[3{E_k}^2 - 9E_k\int_\Omega u_kR^2 + \frac{A_k}{B_k}E_k\int_\Omega 2u_kR\\ &- 2A_k\int_\Omega u_kR^2 + \mathcal{O}(\varepsilon^2\|R\|_{2+p}^{2+p}) + \mathcal{O}(\|R\|^3)\Big],\end{aligned}$$

which in turn implies

$$\begin{aligned}\frac{A}{B}E - \frac{A_k}{B_k}E_k =& \frac{3{E_k}^2 - 9E_k\int_\Omega u_kR^2 + \frac{A_k}{B_k}E_k\int_\Omega 2u_kR}{B_k}\\ &- \frac{A_k}{B_k}\int_\Omega 2u_kR^2 + \mathcal{O}(\varepsilon\|R\|_{2+p}^{2+p}) + \mathcal{O}(\varepsilon^{-1}\|R\|^3).\end{aligned}$$

Using Lemma 6.1 again, this implies

$$\begin{aligned}\frac{A}{B}E - \frac{A_k}{B_k}E_k =& \frac{3{E_k}^2 + \frac{A_k}{B_k}E_k\int_\Omega 2u_kR}{B_k} - \frac{A_k}{B_k}\int_\Omega 2u_kR^2\\ &+ \mathcal{O}(\varepsilon\|R\|_{2+p}^{2+p}) + \mathcal{O}(\varepsilon^{-1}\|R\|^3).\end{aligned} \tag{6.8}$$

The term $-\frac{A_k}{B_k}\int_\Omega 2u_kR^2$ is harmless since it will be handled by the spectrum estimate Lemma 6.3. Let us analyze the fraction which is the worst term. For $M > 1$ to be selected later, define $\|R\|_{\mathcal{T}}$, $\|R\|_{\mathcal{T}^c}$, the L^2 norms of R in the tube $\mathcal{T} := \{(x,t) : |d(x,t)| \le M\varepsilon\}$, the complement of the tube respectively:

$$\|R\|_{\mathcal{T}}^2 := \int_{\{|d(x,t)|\le M\varepsilon\}} R^2(x,t)\,dx, \quad \|R\|_{\mathcal{T}^c}^2 := \int_{\{|d(x,t)|\ge M\varepsilon\}} R^2(x,t)\,dx.$$

Observe that the $\mathcal{O}(\varepsilon)$ size of the tube allows to write

$$\left|\int_{\mathcal{T}} u_kR\right| \le \left(\int_{\mathcal{T}} {u_k}^2\right)^{1/2}\left(\int_{\mathcal{T}} R^2\right)^{1/2} \le C\sqrt{\varepsilon}\|R\|_{\mathcal{T}}.$$

Hence, using Lemma 6.1, cutting $\int_\Omega = \int_{\mathcal{T}} + \int_{\mathcal{T}^c}$, we get

$$\begin{aligned}\left|\frac{A_k}{B_k}E_k\int_\Omega 2u_kR\right| \le& C\varepsilon|E_k|(\sqrt{\varepsilon}\|R\|_{\mathcal{T}} + \|R\|_{\mathcal{T}^c})\\ \le& C\varepsilon\sqrt{\varepsilon}|E_k|\,\|R\|_{\mathcal{T}} + C\varepsilon^{2/5}{E_k}^2 + C\varepsilon^{8/5}\|R\|_{\mathcal{T}^c}^2.\end{aligned}$$

As a result

$$\begin{aligned}\frac{3{E_k}^2 + \frac{A_k}{B_k}E_k\int_\Omega 2u_kR}{B_k} &\geq \frac{(3-C\varepsilon^{2/5}){E_k}^2 - C\varepsilon\sqrt{\varepsilon}|E_k|\,\|R\|_{\mathcal{T}}}{B_k} - C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c}\\ &\geq \frac{{E_k}^2 - C\varepsilon\sqrt{\varepsilon}|E_k|\,\|R\|_{\mathcal{T}}}{B_k} - C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c},\end{aligned} \tag{6.9}$$

for small $\varepsilon > 0$. Now, observe that

$${E_k}^2 - C\varepsilon\sqrt{\varepsilon}|E_k|\,\|R\|_{\mathcal{T}} \geq \begin{cases} 0 & \text{if } |E_k| \geq C\varepsilon\sqrt{\varepsilon}\|R\|_{\mathcal{T}} \\ -C^2\varepsilon^3\|R\|^2_{\mathcal{T}} & \text{if } |E_k| \leq C\varepsilon\sqrt{\varepsilon}\|R\|_{\mathcal{T}}. \end{cases} \tag{6.10}$$

Remark 6.2. The above inequality is the crucial one. One can interpret it as follows. Following [8, Proposition 2], we understand that E_k behaves like the integral on the hypersurface Γ_t:

$$\varepsilon\int_{d(x,t)=0} R(x,t)\,d\sigma.$$

If $|E_k| = \left|\int_\Omega(1-{u_k}^2)R\right|$ is large w.r.t. $\mathcal{O}(\varepsilon\sqrt{\varepsilon}\|R\|_{\mathcal{T}})$ then ${E_k}^2 - C\varepsilon\sqrt{\varepsilon}|E_k|\|R\|_{\mathcal{T}} \geq 0$, which has the good sign to control the L^2 norm of R. In other words, if the error "intends" at concentrating on the hypersurface, the situation is quite favorable. On the other hand, if $|E_k| = \left|\int_\Omega(1-{u_k}^2)R\right|$ is small w.r.t. $\mathcal{O}(\varepsilon\sqrt{\varepsilon}\|R\|_{\mathcal{T}})$ then we get the negative control $-\mathcal{O}(\varepsilon^2\|R\|^2_{\mathcal{T}})$ (after dividing by B_k) which is enough for the Gronwall's argument to work.

Putting together (6.7), (6.8), (6.9), (6.10) and $B_k = 2\sqrt{2}|U|\varepsilon + \mathcal{O}(\varepsilon^2)$, we come to the conclusion that

$$\begin{aligned}\Lambda \geq &-\frac{A_k}{B_k}\int_\Omega 2u_kR^2 - C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c} - C\varepsilon^2\|R\|^2_{\mathcal{T}}\\ &+ \mathcal{O}(\varepsilon^{k+\frac{3}{2}})\|R\| + \mathcal{O}(\varepsilon\|R\|^{2+p}_{2+p}) + \mathcal{O}(\varepsilon^{-1}\|R\|^3).\end{aligned} \tag{6.11}$$

6.2. **Proof of Theorem 2.3.** Equipped with the accurate estimate (6.11), we can now conclude the proof of the error estimate by following the lines of [14]. Combining (6.4) with (6.11) and using the interpolation inequality $\|R\|^{2+p}_{2+p} \leq C\|R\|^p\|\nabla R\|^2$, $\|R\|_{\mathcal{T}} \leq \|R\|$ and $\|R\| = \mathcal{O}(\varepsilon^2)$ (thanks to the definition of t_ε), we discover

$$\begin{aligned}\|R\|\frac{d}{dt}\|R\| \leq &-\int_\Omega|\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega\left(f'(u_k) + \frac{A_k}{B_k}2u_k\right)R^2\\ &+ \frac{1}{\varepsilon^2}2C_1\|R\|^p\|\nabla R\|^2 + \frac{1}{\varepsilon^2}C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c}\\ &+ C\|R\|^2 + \mathcal{O}(\varepsilon^{k-\frac{1}{2}})\|R\|.\end{aligned}$$

Since $\varepsilon^2\left(-\int_\Omega|\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega(f'(u_k) + \frac{A_k}{B_k}2u_k)R^2\right) \leq -\varepsilon^2\|\nabla R\|^2 + C\|R\|^2$, we get

$$\begin{aligned}\|R\|\frac{d}{dt}\|R\| \leq &(1-\varepsilon^2)\left(-\int_\Omega|\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega\left(f'(u_k) + \frac{A_k}{B_k}2u_k\right)R^2\right)\\ &-\varepsilon^2\|\nabla R\|^2 + \frac{1}{\varepsilon^2}2C_1\|R\|^p\|\nabla R\|^2\\ &+ \frac{1}{\varepsilon^2}C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c} + C\|R\|^2 + \mathcal{O}(\varepsilon^{k-\frac{1}{2}})\|R\|\\ \leq &(1-\varepsilon^2)\left(-\int_\Omega|\nabla R|^2 + \frac{1}{\varepsilon^2}\int_\Omega\left(f'(u_k) + \frac{A_k}{B_k}2u_k\right)R^2\right)\\ &+ \frac{1}{\varepsilon^2}C\varepsilon^{3/5}\|R\|^2_{\mathcal{T}^c} + C\|R\|^2 + \mathcal{O}(\varepsilon^{k-\frac{1}{2}})\|R\|,\end{aligned} \tag{6.12}$$

in view of the definition of t_ε in (6.5). In the above inequality, let us write $\int_\Omega = \int_{\mathcal{T}} + \int_{\mathcal{T}^c}$. In the complement of the tube, observe that

$$\int_{\mathcal{T}^c} \left(f'(u_k) + \frac{A_k}{B_k} 2u_k + C\varepsilon^{3/5} \right) R^2 = \int_{\{|d(x,t| \geq M\varepsilon\}} \left(f'(u_k) + \mathcal{O}(\varepsilon^{3/5}) \right) R^2,$$

is nonpositive if $M > 0$ is large enough; this follows from the form of the constructed u_k in Section 5 — roughly speaking we have $u_k(x,t) = \theta_0 \left(\frac{d(x,t) + \mathcal{O}(\varepsilon^2)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2)$— $\theta_0(\pm\infty) = \pm 1$ and $f'(\pm 1) < 0$. As a result we collect

$$\begin{aligned} \|R\| \frac{d}{dt} \|R\| \leq & (1 - \varepsilon^2) \left(- \int_{\mathcal{T}} |\nabla R|^2 + \frac{1}{\varepsilon^2} \int_{\mathcal{T}} \left(f'(u_k) + \frac{A_k}{B_k} 2u_k \right) R^2 \right) \\ & + C\|R\|^2 + \mathcal{O}(\varepsilon^{k - \frac{1}{2}}) \|R\|. \end{aligned} \tag{6.13}$$

In some sense, the problem now reduces to a local estimate since the linearized operator $-\Delta - \varepsilon^{-2}(f'(u_k) + \frac{A_k}{B_k} 2u_k)$ arises when studying the local unbalanced Allen-Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} \left(f(u_\varepsilon) - \frac{A_k}{B_k} (1 - {u_\varepsilon}^2) \right),$$

whose singular limit is "mean curvature plus a forcing term" (see, among others, [2]). To conclude we need a spectrum estimate of the unbalanced linearized operator around the approximate solution u_k, namely $-\Delta - \varepsilon^{-2}(f'(u_k) + \frac{A_k}{B_k} 2u_k)$. This directly follows from the result of [13] for the balanced case. For related results on the spectrum of linearized operators for the Allen-Cahn equation or the Cahn-Hilliard equation, we also refer to [7], [5, 6], [24].

Lemma 6.3 (Spectrum of the unbalanced linearized operator around u_k [13])**.** *There is $C^* > 0$ such that*

$$- \int_{\mathcal{T}} |\nabla R|^2 + \frac{1}{\varepsilon^2} \int_{\mathcal{T}} \left(f'(u_k) + \frac{A_k}{B_k} 2u_k \right) R^2 \leq C^* \int_{\mathcal{T}} R^2,$$

for all $0 < t \leq T$, all $0 < \varepsilon \leq 1$, all $R \in H^1(\Omega)$ with $\int_\Omega R = 0$.

Proof. Observe that

$$u_k(x,t) = \begin{cases} \theta_0 \left(\frac{d_k(x,t)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2) & \text{if } |d(x,t)| \leq \sqrt{\varepsilon} \\ \pm 1 + \mathcal{O}(\varepsilon^{k+1}) & \text{if } |d(x,t)| \geq \sqrt{\varepsilon}. \end{cases}$$

Lemma 6.1 yields $\frac{A_k}{B_k} = \varepsilon \frac{\alpha(t)}{\beta} + \mathcal{O}(\varepsilon^2)$ so that we can write $f'(u_k) + \frac{A_k}{B_k} 2u_k = f'(\overline{u_k})$, for some $\overline{u_k}$ such that

$$\overline{u_k}(x,t) = \begin{cases} \theta_0 \left(\frac{d_k(x,t)}{\varepsilon} \right) - \varepsilon \frac{\alpha(t)}{3\beta} \theta_1 \left(\frac{d_k(x,t)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2) & \text{if } |d(x,t)| \leq \sqrt{\varepsilon} \\ \pm 1 + \mathcal{O}(\varepsilon) & \text{if } |d(x,t)| \geq \sqrt{\varepsilon}, \end{cases}$$

where $\theta_1 \equiv 1$. In particular $\int_{\mathbb{R}} \theta_1 ({\theta_0}')^2 f''(\theta_0) = \int_{\mathbb{R}} ({\theta_0}')^2 f''(\theta_0) = 0$ (odd function) so that $\overline{u_k}$ has the correct shape for [13] to apply: see [4, formula (3.8) and proof of Theorem 5.1], [14, formula (16)] or [22, Section 4] for very related arguments. Details are omitted. □

Combining the above lemma and (6.13), we end up with

$$\frac{d}{dt} \|R\| \leq C\|R\| + C\varepsilon^{k - \frac{1}{2}}.$$

The Gronwall's lemma then implies that, for all $0 \leq t \leq t_\varepsilon$,

$$\|R(\cdot, t)\| \leq (\|R(\cdot, 0)\| + \varepsilon^{k - \frac{1}{2}}) e^{Ct_\varepsilon} = \mathcal{O}(\varepsilon^{k - \frac{1}{2}}),$$

in view of (6.1). Since $k - \frac{1}{2} > \frac{4}{p}$, this shows that $t_\varepsilon = T$ and that the estimate $\mathcal{O}(\varepsilon^{4/p})$ is actually improved to $\mathcal{O}(\varepsilon^{k-\frac{1}{2}})$. This completes the proof of Theorem 2.3. □

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