A functional inequalities approach for the field-road diffusion model with symmetric nonlinear exchanges

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Abstract

In this note, we consider the so-called field-road diffusion model in a bounded domain, consisting of two parabolic PDEs posed on sets of different dimensions and coupled through (symmetric) *nonlinear* exchange terms. We propose a new and rather direct functional inequalities approach to prove the exponential decay of a relative entropy, and thus the convergence of the solution towards the stationary state selected by the total mass of the initial datum.

1. Introduction

In this note, we are interested in the large time behavior of the solution $(v, u) = (v(t, x, y), u(t, x))$ to the so-called *field-road diffusion model*

$$\partial_t v = d\Delta v, \qquad t > 0,\ x \in \omega,\ y \in (0, L), \tag{1.1a}$$

$$-\, d\,\partial_y v|_{y=0} = \alpha(\mu_0 u^\beta - \nu_0 (v|_{y=0})^\alpha), \qquad t > 0,\ x \in \omega, \tag{1.1b}$$

$$\partial_t u = D\Delta u + \beta(\nu_0 (v|_{y=0})^\alpha - \mu_0 u^\beta), \qquad t > 0,\ x \in \omega, \tag{1.1c}$$

$$\frac{\partial u}{\partial n'} = 0, \qquad t > 0,\ x \in \partial\omega, \tag{1.1d}$$

$$\frac{\partial v}{\partial n} = 0, \qquad t > 0,\ x \in \partial\omega,\ y \in (0, L), \text{ and } x \in \omega,\ y = L, \tag{1.1e}$$

supplemented with an initial condition $(v_0, u_0) \in L^\infty(\Omega) \times L^\infty(\omega)$. Here, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded cylinder (the field) of the form

$$\Omega = \omega \times (0, L), \quad \omega \text{ a bounded convex and open set of } \mathbb{R}^{N-1} \text{ (the road)},\ L > 0.$$

The unknowns v and u correspond to the densities of individuals, respectively in the field Ω and on the road ω; d and D are the (positive) diffusion coefficients in the field and on the road. Obviously, Δv has to be understood as $\Delta_x v + \partial_{yy} v$, while Δu has to be understood as $\Delta_x u$. For u we impose the zero Neumann boundary conditions on the boundary $\partial\omega$ (n' denotes the unit outward normal vector to $\partial\omega$). For v, we impose the zero Neumann boundary conditions on the lateral boundary $\partial\omega \times (0, L)$ and on the upper boundary $\omega \times \{L\}$ (n denotes the unit outward normal vector to $\partial\Omega$). On the lower boundary $\omega \times \{0\}$, the exchanges between the field and the road correspond to the value of the outward flux of v given by (1.1b), where $\mu_0 > 0$ and $\nu_0 > 0$ are transfer coefficients. The conservation of the total amount of individuals enforce the zeroth-order term in (1.1c), linking the field and the road equations; they are the core of the model and are here assumed to be *nonlinear*, namely $(\alpha, \beta) \in [1, +\infty)^2 \setminus \{(1, 1)\}$.

The *reaction-diffusion* field-road model was introduced by Berestycki, Roquejoffre and Rossi (2013a,b, 2016a,b) as a model for the spreading of diseases or invasive species in presence of networks with fast propagation (typically $D > d$). We refer to the introduction in Alfaro and Chainais-Hillairet (2025) for more details and references.

Very recently, a series of works has focused on the *purely diffusive* field-road system: the fundamental solution was obtained in Alfaro, Ducasse and Tréton (2023), the PDE model was retrieved as the hydrodynamic limit of a particle system in Alfaro, Mourragui and Tréton (2025). The system (1.1) with linear exchanges ($\alpha = \beta = 1$) was studied in

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Alfaro and Chainais-Hillairet (2025): in both the continuous and the discrete settings, the dissipation of a quadratic entropy is proved, the main tool being an adapted Poincaré-Wirtinger inequality, see Lemma 3.1 below.

Note that (1.1) stands in the class of *volume-surface* systems, considered e.g. in Glitzky and Mielke (2013), Casteras, Monsaingeon and Santambrogio (2025) through gradient flow techniques. More connected to our approach are the works Fellner, Latos and Tang (2018), Egger, Fellner, Pietschmann and Tang (2018), and the references therein. In Fellner et al. (2018), nonlinear exchanges are considered and the long time behavior is studied thanks to the (logarithmic) Boltzmann entropy. The proof is rather lengthy and intricate and our goal here is to provide a more direct approach.

To do so, in this note, we consider the case of *symmetric* nonlinear exchanges, namely

$$\alpha = \beta > 1. \tag{1.2}$$

For simplicity we use the shortcuts $\mu = \alpha\mu_0$, $\nu = \alpha\nu_0$ so that the right hand side of the equation (1.1b) is reduced to $\mu u^\alpha - \nu(v|_{y=0})^\alpha$. In this framework, we consider a (power) Tsallis entropy and rely on two distinct functional inequalities: a Poincaré-Wirtinger inequality adapted to the field-road coming from Alfaro and Chainais-Hillairet (2025) and a Beckner type inequality Beckner (1989) coming from Chainais-Hillairet, Jüngel and Schuchnigg (2016). We believe the arguments become transparent and, furthermore, this approach can be transferred to the design of a numerical scheme preserving the main properties of the system, see Alfaro and Chainais-Hillairet (2025) for the linear case.

We hope the method to be adapted to even more complex situations. In particular we aim at addressing the issue of *nonsymmetric* exchanges, in the sense that $\alpha \neq \beta$, considered in Fellner et al. (2018). This still requires an improvement of our method.

This note is organized as follows. In Section 2 we present the result, which is proved in Section 3 and completed by numerical explorations in Section 4.

2. Setting of the result

We start with some basic facts. We consider $v_0 \in L^\infty(\Omega)$, $u_0 \in L^\infty(\omega)$, both nonnegative and not simultaneously trivial. As a result, the total mass is initially positive

$$M_0 := \int_\Omega v_0(x,y)\,dxdy + \int_\omega u_0(x)\,dx > 0.$$

As for the definition, existence and uniqueness of the weak solution to the Cauchy problem, we may use the approach of Egger et al. (2018) or Alfaro and Chainais-Hillairet (2025) (based on a single spatial integration by parts) but, for the sake of conciseness, we borrow the results of Section 2 in Fellner et al. (2018) (based on a spatial integration by part *and* a time integration by part). We denote $(v = v(t,x,y), u = u(t,x))$ the weak solution starting from $(v_0 = v_0(x,y), u_0 = u_0(x))$ and note that it is immediately smooth thanks to parabolic regularity.

Also, since the model can be thought as a *cooperative* system, it enjoys a comparison principle. We may refer to Fellner et al. (2018) (for weak solutions) or to Berestycki et al. (2013b) (for regular solutions). Hence, since the initial data are nonnegative and bounded, we may choose $C_1 \geq \|v_0\|_{L^\infty}$, $C_2 \geq \|u_0\|_{L^\infty}$ with $\mu C_2^\alpha = \nu C_1^\alpha$ (so that (C_1, C_2) is a solution to the system) and deduce from the comparison principle that both v and u are nonnegative and uniformly bounded, namely $0 \leq v \leq C_1$, $0 \leq u \leq C_2$.

The total mass of the system $\int_\Omega v(t,x,y)\,dxdy + \int_\omega u(t,x)\,dx$ is constant, namely

$$\int_\Omega v(t,x,y)\,dxdy + \int_\omega u(t,x)\,dx = M_0, \quad \forall t > 0. \tag{2.1}$$

The unique constant steady-state (v_∞, u_∞) with mass M_0 is given by

$$\nu v_\infty^\alpha = \mu u_\infty^\alpha, \quad |\Omega| v_\infty + |\omega| u_\infty = M_0. \tag{2.2}$$

Note that the existence of nonconstant steady-states is excluded later, as a direct consequence of Theorem 2.1.

We apply a relative entropy method as presented for instance in the book by Jüngel (2016). Define

$$\Phi(s) := \frac{s^{\alpha+1} - (\alpha+1)s}{\alpha} + 1,$$

which satisfies $\Phi'' > 0$, $\Phi'(1) = 0$, $\Phi(1) = 0$. We define a nonnegative entropy, relative to the steady-state (v_∞, u_∞), by

$$\mathcal{H}(t) := \int_\Omega v_\infty \Phi\left(\frac{v(t,x,y)}{v_\infty}\right) dxdy + \int_\omega u_\infty \Phi\left(\frac{u(t,x)}{u_\infty}\right) dx, \tag{2.3}$$

which, using (2.1) and (2.2), can be recast

$$\mathcal{H}(t) = \frac{1}{\alpha v_\infty^\alpha} \int_\Omega \left(v^{\alpha+1}(x,y) - v_\infty^{\alpha+1}\right) dxdy + \frac{1}{\alpha u_\infty^\alpha} \int_\omega \left(u^{\alpha+1}(x) - u_\infty^{\alpha+1}\right) dx. \tag{2.4}$$

The main result then writes as follows.

Theorem 2.1 (Exponential decay of entropy). *Assume $\alpha = \beta > 1$. Let $v_0 \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\omega)$ be both nonnegative and not simultaneously trivial. Let $(v = v(t,x,y), u = u(t,x))$ be the solution to (1.1) starting from $(v_0 = v_0(x,y), u_0 = u_0(x))$, and (v_∞, u_∞) the associated steady-state defined by (2.2). Then the entropy defined by (2.3) decays exponentially, namely*

$$0 \leq \mathcal{H}(t) \leq \mathcal{H}(0) e^{-\lambda t}, \quad \forall t > 0, \tag{2.5}$$

for some positive $\lambda = \lambda(N, \Omega, \mu, \nu, d, D, v_0, u_0, \alpha)$.

As a by-product,

$$\|v - v_\infty\|_{L^{\alpha+1}(\Omega)}^{\alpha+1} + \|u - u_\infty\|_{L^{\alpha+1}(\omega)}^{\alpha+1} \leq M e^{-\lambda t}, \quad \forall t > 0, \tag{2.6}$$

for some positive $M = M(N, \Omega, \mu, \nu, d, D, v_0, u_0, \alpha)$.

3. A functional inequalities approach

We prove Theorem 2.1. In the sequel, the notation $\mathcal{A} \lesssim \mathcal{B}$ means that $\mathcal{A} \leq C\mathcal{B}$ for some positive constant $C = C(N, \Omega, \omega, \mu, \nu, d, D, v_0, u_0, \alpha)$.

By differentiating expression (2.4) with respect to time, using the equations in (1.1) and integration by parts we reach

$$\frac{\alpha}{\alpha+1}\frac{d}{dt}\mathcal{H}(t) = \frac{1}{v_\infty^\alpha}\left(\int_\omega (\mu u^\alpha - \nu(v|_{y=0})^\alpha) v^\alpha - d\int_\Omega \alpha \nabla v \cdot v^{\alpha-1}\nabla v\right) + \frac{1}{u_\infty^\alpha}\left(-D\int_\omega \alpha \nabla u \cdot u^{\alpha-1}\nabla u + \int_\omega (\nu(v|_{y=0})^\alpha - \mu u^\alpha) u^\alpha\right).$$

Thanks to (2.2) we can gather the two non gradient terms and obtain

$$\frac{d}{dt}\mathcal{H}(t) \lesssim -\int_\Omega |\nabla(v^{\frac{\alpha+1}{2}})|^2 - \int_\omega |\nabla(u^{\frac{\alpha+1}{2}})|^2 - \int_\omega \left(\nu(v|_{y=0})^\alpha - \mu u^\alpha\right)^2 =: -\mathcal{D}(t). \tag{3.1}$$

We now take advantage of the adapted Poincaré-Wirtinger inequality developed in Alfaro and Chainais-Hillairet (2025). To do so, for $\ell > 0$, we "enlarge" $\Omega = \omega \times (0, L)$ to $\Omega^+ = \omega \times (-\ell, L)$. We denote $\Omega_\ell = \omega \times (-\ell, 0)$ the so-called thickened road. We work with

$$d\rho = \left(\frac{v_\infty}{M_0}\mathbf{1}_\Omega(x,y) + \frac{1}{\ell}\frac{u_\infty}{M_0}\mathbf{1}_{\Omega_\ell}(x,y)\right) dxdy, \tag{3.2}$$

which is a probability measure as can be checked thanks to (2.2), and with

$$f(x,y) = \left(\frac{v(x,y)}{v_\infty}\right)^\alpha \mathbf{1}_\Omega(x,y) + \left(\frac{u(x)}{u_\infty}\right)^\alpha \mathbf{1}_{\Omega_\ell}(x,y), \quad (x,y) \in \Omega^+, \tag{3.3}$$

where we have omitted to write the t variable. By strictly reproducing the proof of (Alfaro and Chainais-Hillairet, 2025, Theorem 1), which was concerned with the case $\alpha = 1$, we obtain the following.

Lemma 3.1 (Adapted Poincaré-Wirtinger inequality)**.** *Defining* $\langle f \rangle := \int_{\Omega^+} f\, d\rho$, *there holds*

$$\|f - \langle f \rangle\|^2_{L^2(\Omega^+, d\rho)} \lesssim \int_\Omega |\nabla(v^\alpha)|^2 + \int_\omega |\nabla(u^\alpha)|^2 + \int_\omega \left(\nu(v|_{y=0})^\alpha - \mu u^\alpha\right)^2. \tag{3.4}$$

Next, we borrow (Chainais-Hillairet et al., 2016, Lemma 7).

Lemma 3.2 (Generalized Beckner inequality II)**.** *For* $0 < q < 2$, $pq \geq 1$, *there holds*

$$\|f\|^{2-q}_{L^q((\Omega^+, d\rho)} \left(\int_{\Omega^+} |f|^q d\rho - \left(\int_{\Omega^+} |f|^{\frac{1}{p}} d\rho \right)^{pq} \right) \lesssim \|f - \langle f \rangle\|^2_{L^2(\Omega^+, d\rho)}. \tag{3.5}$$

Now, observe that since v and u are uniformly bounded there holds

$$|\nabla(u^\alpha)| \lesssim |\nabla(u^{\frac{\alpha+1}{2}})|, \quad |\nabla(v^\alpha)| \lesssim |\nabla(v^{\frac{\alpha+1}{2}})|.$$

It therefore follows from Lemma 3.1 and Lemma 3.2 (with $q = \frac{\alpha+1}{\alpha}$, $p = \alpha$) that

$$\mathcal{D}(t) \gtrsim \|f\|^{\frac{\alpha-1}{\alpha}}_{L^{\frac{\alpha+1}{\alpha}}(\Omega^+, d\rho)} \left(\int_{\Omega^+} |f|^{\frac{\alpha+1}{\alpha}} d\rho - \left(\int_{\Omega^+} |f|^{\frac{1}{\alpha}} d\rho \right)^{\alpha+1} \right).$$

Now using (3.3), (3.2), and (2.1), we see that

$$\begin{aligned} \int_{\Omega^+} |f|^{\frac{\alpha+1}{\alpha}} d\rho - \left(\int_{\Omega^+} |f|^{\frac{1}{\alpha}} d\rho \right)^{\alpha+1} &= \int_\Omega \left(\frac{v}{v_\infty} \right)^{\alpha+1} \frac{v_\infty}{M_0} dx dy + \int_\omega \left(\frac{u}{u_\infty} \right)^{\alpha+1} \frac{u_\infty}{M_0} dx - 1 \\ &= \frac{1}{M_0 v_\infty^\alpha} \int_\Omega \left(v^{\alpha+1} - v_\infty^{\alpha+1} \right) + \frac{1}{M_0 u_\infty^\alpha} \int_\omega \left(u^{\alpha+1} - u_\infty^{\alpha+1} \right), \end{aligned}$$

by using the second relation in (2.2). Next, by Jensen's inequality,

$$\|f\|^{\frac{1}{\alpha}}_{L^{\frac{\alpha+1}{\alpha}}(\Omega^+, d\rho)} = \left(\int_{\Omega^+} f^{\frac{\alpha+1}{\alpha}} d\rho \right)^{\frac{1}{\alpha+1}} \geq \int_{\Omega^+} f^{\frac{1}{\alpha}} d\rho = 1.$$

As a result, we end up with

$$\mathcal{D}(t) \gtrsim \frac{1}{M_0 v_\infty^\alpha} \int_\Omega \left(v^{\alpha+1} - v_\infty^{\alpha+1} \right) + \frac{1}{M_0 u_\infty^\alpha} \int_\omega \left(u^{\alpha+1} - u_\infty^{\alpha+1} \right) = \frac{\alpha}{M_0} \mathcal{H}(t). \tag{3.6}$$

In view of (3.6) and (3.1), we collect $\frac{d}{dt}\mathcal{H}(t) \lesssim -\mathcal{H}(t)$, which proves (2.5).

Last the decay of the $L^{\alpha+1}$ norm, namely (2.6), follows from (2.3) and the fact that $|1-s|^{\alpha+1} \leq s^{\alpha+1} - (\alpha+1)s + \alpha = \alpha\Phi(s)$ for all $s \geq 0$.

Theorem 2.1 is proved. □

4. Numerical experiments

Our aim in this section is to illustrate the exponential decay of the relative entropy $\mathcal{H}$ defined by (2.3), as stated in Theorem 2.1 for the nonlinear field-road model (1.1) with symmetric exchanges ($\alpha = \beta$). We will also investigate the behaviour of the similar relative entropy in the case with nonsymmetric exchanges ($\alpha \neq \beta$). In order to do some numerical investigations, we use a two-point flux approximation (TPFA) finite volume scheme, with a backward in time Euler method, as introduced in Alfaro and Chainais-Hillairet (2025) for $\alpha = \beta = 1$. Due to the nonlinear exchanges, the scheme consists in a nonlinear system of equations at each time step, which is solved using Newton's method.

For the numerical experiments, we consider that the one-dimensional road is $\omega = (-2L, 2L)$ and the two-dimensional field is $\Omega = \omega \times (0, L)$ with $L = 20$. The value of the kinetical parameters μ_0 and ν_0 are $\mu_0 = 1$, $\nu_0 = 5$. The diffusion parameters, in the field and in the road, are respectively $d = 1$, $D = 1$. We consider two test cases already proposed in Alfaro and Chainais-Hillairet (2025) and defined in Table 1. In both test cases, the individuals

Table 1
Presentation of the test cases used for the numerical experiments.

	Test case 1	Test case 2
$v_0(x, y)$	$100 \cdot \mathbf{1}_{[-10,-7.5]\cup[-5,-2.5]\cup[2.5,5]\cup[7.5,10]}(x) \cdot \mathbf{1}_{[7.5,10]}(y)$	$150 \cdot \mathbf{1}_{[-10,-7.5]\cup[-5,-2.5]\cup[2.5,5]\cup[7.5,10]}(x) \cdot \mathbf{1}_{[8.75,10]}(y)$
$u_0(x)$	0	$62.5 \cdot \mathbf{1}_{[-10,-7.5]\cup[-5,-2.5]\cup[2.5,5]\cup[7.5,10]}(x)$

are scattered in the field, but the road is empty in Test case 1, while there are some individuals scattered in the road in Test case 2.

Figure 1 shows the long-time behaviour of the relative entropy $\mathcal{H}$ for both test cases in the symmetric case. Figure 2 shows the same evolution but in the nonsymmetric case. In the symmetric as in the nonsymmetric case, we observe that the decay of the relative entropy in time is exponential. Moreover, we observe that the decay rate seems to be independent of the values of α and β. As confirmed by further simulations (not shown here) where we drastically change the initial mass, the decay rate also seems to be independent of the initial condition.

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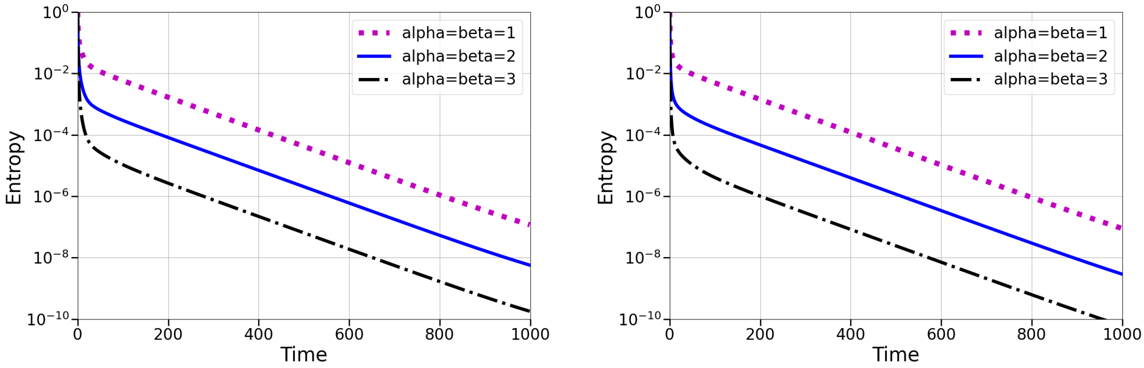

Figure 1: Exponential decay of the relative entropy in the symmetric case, $\alpha = \beta$, for Test case 1 (left) and Test case 2 (right).

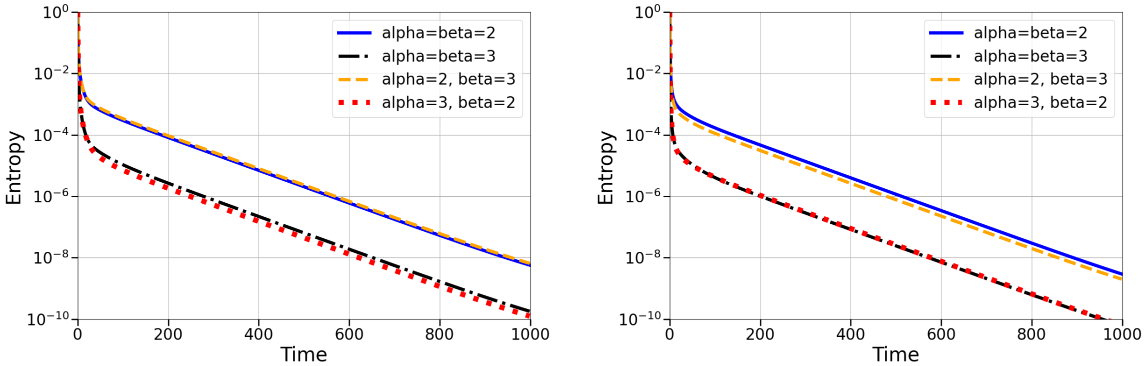

Figure 2: Comparison of the decay of the relative entropy in the symmetric and nonsymmetric cases, for Test case 1 (left) and Test case 2 (right).