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ON THE VALIDITY OF FORMAL ASYMPTOTIC EXPANSIONS IN ALLEN-CAHN EQUATION AND FITZHUGH-NAGUMO SYSTEM WITH GENERIC INITIAL DATA

Matthieu Alfaro

Univ. Montpellier 2, I3M, UMR CNRS 5149,
CC051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France,

Hiroshi Matano

Univ. of Tokyo, Graduate School of Mathematical Sciences,
3-8-1 Komaba, Tokyo 153-8914, Japan.

Dedicated to Prof. Avner Friedman on the occasion of his eightieth birthday

Abstract. Formal asymptotic expansions have long been used to study the singularly perturbed Allen-Cahn type equations and reaction-diffusion systems, including in particular the FitzHugh-Nagumo system. Despite their successful role, it has been largely unclear whether or not such expansions really represent the actual profile of solutions with rather general initial data. By combining our earlier result and known properties of eternal solutions of the Allen-Cahn equation, we prove validity of the principal term of the formal expansions for a large class of solutions.

1. **Introduction.** In this paper, we study the behavior of solution u^ε of an Allen-Cahn type equation of the form

$$(\mathrm{P}^\varepsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon g^\varepsilon(x,t,u)) & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega\,, \end{cases}$$

and also that of a reaction-diffusion system of the form

$$(\mathrm{RD}^\varepsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\left(f(u) + \varepsilon f_1(u,v) + \varepsilon^2 f_2^\varepsilon(u,v)\right) & \text{in } \ \Omega \times (0,\infty) \\ v_t = D\Delta v + h(u,v) & \text{in } \ \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \ \Omega \\ v(x,0) = v_0(x) & \text{in } \ \Omega\,, \end{cases}$$

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where $f(u), g^\varepsilon(x,t,u), f_1(u,v)$ and $f_2^\varepsilon(u,v)$ satisfy the conditions to be specified later, and ε is a positive parameter. A typical example of (RD$^\varepsilon$) is the FitzHugh-Nagumo system:

$$(\mathrm{FHN}^\varepsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\left(f(u) - \varepsilon f_1(u) - \varepsilon v\right) & \text{in } \Omega \times (0,\infty) \\ v_t = D\Delta v + \alpha u - \beta v & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ v(x,0) = v_0(x) & \text{in } \Omega\,. \end{cases}$$

It is well-known that the solution $u^\varepsilon(x,t)$ of the above systems develops a steep transition layer, which converges to a "sharp interface" as $\varepsilon \to 0$. To study such a sharp interface limit, formal asymptotic expansions of u^ε are commonly used to discover, formally, the law of motion of the limit interface. Then, based on these expansions, one can construct sub- and super-solutions or use some approximation argument to prove the convergence of the transition layer — or the front — to the sharp interface, thereby establishing rigorously that the limit motion law agrees with what is anticipated from the formal asymptotics.

However, this standard approach only tells us that the transition layer of u^ε is confined within a relatively narrow zone — of thickness $o(1)$ or sometimes even $\mathcal{O}(\varepsilon)$— around the limit interface, but it does not say much about whether or not the actual transition layer really possesses a robust profile that matches the formal asymptotics. Known answers to this question are mainly concerned with solutions whose initial data already has a well-developed transition layer. The case of more general solutions has largely been unexplored. Our goal is to provide an affirmative answer in this direction: we shall prove that, for ε sufficiently small, the solution u^ε — with rather general initial data — of both (P$^\varepsilon$) and (RD$^\varepsilon$) possesses a profile that agrees with the principal term of the formal expansion.

1.1. **Notation and assumptions.** The notation and assumptions stated below strictly follow those in [2].

In problems (P$^\varepsilon$) and (RD$^\varepsilon$) above, Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and ν denotes the outward unit normal vector to $\partial\Omega$. The nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = \alpha_\pm$. More precisely we assume that f is C^2 and has exactly three zeros $\alpha_- < a < \alpha_+$ such that

$$f'(\alpha_\pm) < 0\,, \quad f'(a) > 0 \qquad \text{(bistable nonlinearity)}, \tag{1}$$

and that

$$\int_{\alpha_-}^{\alpha_+} f(u)\,du = 0 \qquad \text{(balanced case)}. \tag{2}$$

In the Allen-Cahn equation (P$^\varepsilon$) we allow the balance of the two stable zeros α_- and α_+ to be slightly broken by the function $-\varepsilon g^\varepsilon(x,t,u)$ defined on $\overline{\Omega} \times [0,\infty) \times \mathbb{R}$. We assume that g^ε is C^2 in x and C^1 in t, u, and that, for any $T > 0$ there exist $\vartheta \in (0,1)$ and $C > 0$ such that, for all $(x,t,u) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$,

$$\varepsilon|\Delta_x g^\varepsilon(x,t,u)| + \varepsilon|g_t^\varepsilon(x,t,u)| + |g_u^\varepsilon(x,t,u)| \leq C\,,$$
$$\|g^\varepsilon(\cdot,\cdot,u)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\bar\Omega\times[0,T])} \leq C\,.$$

Moreover, we assume that there exists a function $g(x,t,u)$ and a constant, which we denote again by C, such that

$$|g^\varepsilon(x,t,u) - g(x,t,u)| \le C\varepsilon\,, \tag{3}$$

for all small $\varepsilon > 0$. In [2], we also assumed $\frac{\partial g^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega \times [0,\infty) \times \mathbb{R}$, though this last condition is only for technical simplicity. Note that these conditions, except the last one, are automatically satisfied if g^ε is smooth and independent of ε.

In the reaction-diffusion system (RD^ε) we assume that $f_1(u,v)$, $f_2^\varepsilon(u,v)$ are C^2 functions and that f_2^ε, along with its derivatives, remain bounded as $\varepsilon \to 0$. We also assume that $D > 0$ and that $h(u,v)$ is a C^2 function such that, for any constants $L, M > 0$, there exists a constant $M_1 \ge M$ such that $h(u,-M_1) \ge 0 \ge h(u,M_1)$ for $|u| \le L$. These conditions enable us to construct a family of invariant rectangles mentioned in Remark 1.

In the FitzHugh-Nagumo system, we assume that $f_1 \in C^2(\mathbb{R})$ and that α and β are given positive constants so that (FHN^ε) becomes a special case of the reaction-diffusion system (RD^ε).

To complete the picture we need to specify conditions on the initial data. We assume that u_0 and v_0 belong to $C^2(\bar\Omega)$. We define the "initial interface" Γ_0 by

$$\Gamma_0 := \{x \in \Omega : \, u_0(x) = a\}\,, \tag{4}$$

and assume that Γ_0 is a $C^{3+\vartheta}$ hypersurface ($0 < \vartheta < 1$) without boundary such that

$$\Gamma_0 \subset\subset \Omega \quad \text{ and } \quad \nabla u_0(x) \cdot n(x,0) \neq 0 \quad \text{if } x \in \Gamma_0\,,$$

$$u_0 > a \quad \text{ in } \Omega_0^+\,, \quad u_0 < a \quad \text{ in } \Omega_0^-\,,$$

where Ω_0^- denotes the region enclosed by Γ_0 and Ω_0^+ the region enclosed between $\partial\Omega$ and Γ_0, and $n(x,0)$ denotes the outward unit normal vector at $x \in \Gamma_0 = \partial\Omega_0^-$. Let us emphasize that we do not assume that the initial data u_0 of u^ε already has well-developed transition layers depending on ε, in which case the validity of the formal expansions is more or less known (see subsection 1.2 for more details).

Remark 1 (Time-global smooth solutions)**.** Under the above assumptions, it is classical that (P^ε) has a uniformly bounded smooth solution u^ε that exists for all $t \ge 0$. As for (RD^ε), the same can be shown for $\varepsilon > 0$ small enough, by using the method of invariant rectangles (see e.g. [2] for details).

1.2. **Known results for the singular limit.** We present here a brief overview of known results. Heuristically, in the very early stage, the diffusion term is negligible compared with the reaction term. Hence, in view of the profile of f, the value of u^ε quickly becomes close to either α_+ or α_- in most part of Ω, creating a steep interface (transition layers) between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$ (*Generation of interface*). Once the balance between diffusion and reaction near the transition layers is established, the interface starts to propagate in a much slower time scale (*Motion of interface*). The interface obeys a certain law of motion, which is to be investigated.

A first step to understand this motion is to use (inner and outer) formal asymptotic expansions of u^ε. This was performed in the pioneering work of Allen and Cahn [3] and, slightly later, in Kawasaki and Ohta [18], who revealed that the interface motion involves curvature effects. Using such arguments, one discovers that the sharp interface limit of (P^ε) obeys the following law of motion:

$$(\mathrm{P}^0) \qquad \begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} g(x,t,r)\,dr & \text{on } \Gamma_t \\ \Gamma_t\big|_{t=0} = \Gamma_0\,, \end{cases}$$

where Γ_t denotes the limit sharp interface at time $t \geq 0$, V_n is the normal velocity of Γ_t in the exterior direction, κ the mean curvature at each point of Γ_t, c_0 a constant determined straightforwardly from f via

$$c_0 := \Big[\sqrt{2}\int_{\alpha_-}^{\alpha_+}(W(s)-W(\alpha_-))^{1/2}ds\Big]^{-1},$$

where $W(s) := -\int_a^s f(r)dr$. As long as the solution Γ_t of (P^0) exists, we denote by Ω_t^- the region enclosed by Γ_t, and by Ω_t^+ the region enclosed between $\partial\Omega$ and Γ_t. Also we define a step function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) := \begin{cases} \alpha_- & \text{in } \Omega_t^- \\ \alpha_+ & \text{in } \Omega_t^+\,, \end{cases} \tag{5}$$

to which u^ε is formally supposed to converge as $\varepsilon \to 0$. As regards (RD^ε), the limit problem is found to be

$$(\mathrm{RD}^0) \qquad \begin{cases} V_n = -(N-1)\kappa - c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} f_1(r,v)\,dr & \text{on } \Gamma_t \\ \tilde{v}_t = D\Delta\tilde{v} + h(\tilde{u},\tilde{v}) & \text{in } \Omega\times(0,\infty) \\ \Gamma_t\big|_{t=0} = \Gamma_0 & \\ \dfrac{\partial\tilde{v}}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,\infty) \\ \tilde{v}(x,0) = v_0(x) & \text{in } \Omega\,, \end{cases}$$

where $\tilde{u}$ is the step function defined in (5). This is a system consisting of an equation of surface motion and a parabolic partial differential equation. Since $\tilde{u}$ is determined straightforwardly from Γ_t, in what follows, by a solution of (RD^0) we mean a pair $(\Gamma,\tilde{v})$.

Remark 2 (Local smooth solutions for the limit problems)**.** Under our assumptions, there exists $T^{max} > 0$ such that (P^0), respectively (RD^0), possesses a unique smooth solution $\Gamma = \cup_{0\leq t<T^{max}}(\Gamma_t\times\{t\})$, resp. $(\Gamma,\tilde{v}) = (\cup_{0\leq t<T^{max}}(\Gamma_t\times\{t\}),\tilde{v})$. For more details we refer to [2] and the references therein, in particular [13], [14], [12]. In the sequel we select any $0 < T < T^{max}$ and work on $[0,T]$.

Numerous efforts have been made to rigorously prove the convergence of (P^ε) and (RD^ε) to (P^0) and (RD^0), respectively. Concerning the Allen-Cahn equation, let us mention the work of de Mottoni and Schatzman [20] (generation of interface via sub- and super-solutions) and [21] (motion of interface via construction of and linearization around an ansatz) or that of Bronsard and Kohn [10] (motion of interface via Γ-convergence). Chen [11, 12] has established an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ error estimate between the location of the actual transition layer and the limit interface, both for scalar equations and systems for rather general initial data. More recently, in [2], the present authors improved this estimate to $\mathcal{O}(\varepsilon)$. More precisely, they show that the solution u^ε develops a steep transition layer within the time scale of $\mathcal{O}(\varepsilon^2|\ln\varepsilon|)$, and that the layer obeys the law of motion that coincides with the formal asymptotic limit (P^0) or (RD^0) within an error margin of $\mathcal{O}(\varepsilon)$. Let us also mention that there are results of much finer error estimates in the literature (see for instance [8]), but those results are concerned with very specific initial data which already have nice transition layers consistent with formal asymptotics (hence dependent on ε).

As mentioned before, in most of the aforementioned works, approximate solutions or sub- and super-solutions are constructed by roughly following the formal expansions. Our goal is to investigate the actual validity of such expansions for solutions u^ε with rather general initial data.

Remark 3 (Viscosity framework)**.** Since the limit problem may develop singularities in finite time, the *classical framework* does not always allow to study the singular limit procedure for all $t \geq 0$. Nevertheless — as far as the Allen-Cahn equation is concerned — following [17], [15] one can define a limit problem for all $t \geq 0$ that generalizes (P^0) in the framework of *viscosity solutions*. In this setting we refer to [16] (convergence of Allen-Cahn equation with prepared initial data to generalized motion by mean curvature), [4], [6] (generalizations), [22, 23], [7], [5] (not well-prepared initial data), [1] (fine convergence rate).

2. Main results.

We start by giving an outline of the formal asymptotic expansions mentioned before. See [2, Section 2] for more details. Let u^ε be the solution of (P^ε), and $\Gamma = \cup_{0\leq t\leq T}(\Gamma_t \times \{t\})$ be the solution of the limit geometric motion problem (P^0). We define the *signed distance function* to Γ by

$$d(x,t) := \begin{cases} -\mathrm{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^- \\ \ \ \mathrm{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+ \,. \end{cases} \tag{6}$$

Then, near Γ, we make a formal inner expansion of the form

$$u^\varepsilon(x,t) = U_0\left(x,t,\frac{d(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{d(x,t)}{\varepsilon}\right) + \cdots \,. \tag{7}$$

Some normalization conditions and matching conditions (with the outer expansion) are also imposed. By plugging the expansion (7) into (P^ε), we discover that $U_0(x,t,z) = U_0(z)$, where $U_0(z)$ is the unique solution (whose existence is guaranteed by the integral condition (2)) of the stationary problem

$$\begin{cases} {U_0}'' + f(U_0) = 0 \\ U_0(-\infty) = \alpha_- \,, \quad U_0(0) = a \,, \quad U_0(\infty) = \alpha_+ \,. \end{cases} \tag{8}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. Next the solvability condition for the equation involving U_1 provides the law of motion (P^0) for the limit interface Γ, which, in turn, determines the term U_1.

It is then natural to wonder if the ansatz

$$u^\varepsilon(x,t) = U_0\left(\frac{d(x,t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{d(x,t)}{\varepsilon}\right) + \cdots$$

is really a good approximation of the profile of the solution u^ε. Note that the convergence results mentioned in subsection 1.2 do not answer this question; indeed those results simply show that the level surface of the solution u^ε

$$\Gamma_t^\varepsilon := \{x \in \Omega : \, u^\varepsilon(x,t) = a\} \tag{9}$$

converges to the sharp interface $(\Gamma_t)_{0\leq t\leq T}$, which is a solution of (P^0), and that

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \begin{cases} \alpha^- & \text{for } x\in\Omega_t^- \\ \alpha^+ & \text{for } x\in\Omega_t^+\,, \end{cases} \tag{10}$$

without clarifying the validity of (7). Our main result Theorem 2.1 below provides a first answer in this direction. In the sequel we define

$$t^\varepsilon := f'(a)^{-1}\varepsilon^2|\ln\varepsilon|\,, \tag{11}$$

which is the time needed for the transition layer of u^ε to become fully well-developed (see Lemma 3.1). We define the *signed distance function associated with* Γ^ε by

$$d^\varepsilon(x,t) := \begin{cases} -\mathrm{dist}(x,\Gamma_t^\varepsilon) & \text{if } u^\varepsilon(x,t) < a \\ \mathrm{dist}(x,\Gamma_t^\varepsilon) & \text{if } u^\varepsilon(x,t) > a\,. \end{cases} \tag{12}$$

Note that this definition of d^ε is consistent with that of d in (6) in view of (10).

Theorem 2.1 (Validity for Allen-Cahn). *Let the assumptions of subsection 1.1 hold (in particular the initial condition u_0 is rather generic). Let u^ε be the smooth solution of Allen-Cahn equation* (P^ε)*. Fix $\mu>1$. Then the following hold.*

(i) *If $\varepsilon>0$ is small enough then, for any $t\in[\mu t^\varepsilon, T]$, the level set Γ_t^ε is a smooth hypersurface and can be expressed as a graph over Γ_t.*

(ii)

$$\lim_{\varepsilon\to 0}\ \sup_{\mu t^\varepsilon\leq t\leq T,\, x\in\bar\Omega}\left|u^\varepsilon(x,t) - U_0\left(\frac{d^\varepsilon(x,t)}{\varepsilon}\right)\right| = 0\,, \tag{13}$$

where d^ε denotes the signed distance function associated with Γ^ε.

(iii) *There exists a family of functions*

$$\theta^\varepsilon : \cup_{0\leq t\leq T}(\Gamma_t\times\{t\}) \to \mathbb{R}\quad (0<\varepsilon<<1) \tag{14}$$

whose L^∞-norms remain bounded as $\varepsilon\to 0$, such that

$$\lim_{\varepsilon\to 0}\ \sup_{\mu t^\varepsilon\leq t\leq T,\, x\in\bar\Omega}\left|u^\varepsilon(x,t) - U_0\left(\frac{d(x,t)-\varepsilon\theta^\varepsilon(p(x,t),t)}{\varepsilon}\right)\right| = 0\,, \tag{15}$$

where d denotes the signed distance function associated with Γ and $p(x,t)$ denotes a point on Γ_t such that dist$(x,\Gamma_t) = \|x-p(x,t)\|$.

Note that $p(x,t)$ is an orthogonal projection of the point x onto Γ_t, which is uniquely defined in a small tubular neighborhood of Γ_t since Γ_t is a smooth solution of (P^0). Note also that the presence of the perturbations $-\varepsilon\theta^\varepsilon(p(x,t),t)$ cannot be avoided since it reflects the small difference between $d(x,t)$ and $d^\varepsilon(x,t)$.

Let us mention that the validity of higher order terms of the formal expansions (for generic solutions) is still unknown.

Our next theorem provides similar estimates for the reaction-diffusion systems.

Theorem 2.2 (Validity for the reaction-diffusion system). *Let the assumptions of subsection 1.1 hold, and let $(u^\varepsilon, v^\varepsilon)$ be the smooth solution of the reaction-diffusion system* $(\mathrm{RD}^\varepsilon)$*. Fix $\mu>1$. Then the same conclusions as in Theorem 2.1 hold, with d being the signed distance function associated with $(\Gamma,\tilde v)$, which is the smooth solution of* (RD^0) *on $[0,T]$.*

3. **Proof of the main results.** The proof of Theorem 2.1 relies on the following two results:

(a) *the level set Γ_t^ε is approximated by the interface Γ_t by order $\mathcal{O}(\varepsilon)$ ([2], see subsection 3.1 of the present paper),*

(b) *any eternal solution that lies between two planar waves is actually a planar wave ([9], see subsection 3.2 of the present paper),*

combined with a rescaling argument.

3.1. **Thickness of the layers: the refined $\mathcal{O}(\varepsilon)$ estimate.** We quote a result which is valid for both (P^ε) and (RD^ε).

Lemma 3.1 ([2, Theorem 1.3 and Theorem 1.11])**.** *Let η be an arbitrary constant satisfying $0 < \eta < \min(a - \alpha_-, \alpha_+ - a)$. Then there exist positive constants ε_0 and C_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $f'(a)^{-1}\varepsilon^2|\ln\varepsilon| = t^\varepsilon \le t \le T$, we have*

$$|u^\varepsilon(x,t) - \alpha_\pm| \le \eta \quad \text{if } x \in \Omega_t^\pm \setminus \mathcal{N}_{C_0\varepsilon}(\Gamma_t)\,, \tag{16}$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega : dist(x, \Gamma_t) < r\}$ denotes the r-neighborhood of Γ_t. This implies in particular that $\Gamma_t^\varepsilon \subset \mathcal{N}_{C_0\varepsilon}(\Gamma_t)$ for all $t^\varepsilon \le t \le T$, hence

$$|d^\varepsilon(x,t) - d(x,t)| \le C_0\varepsilon \quad \text{for all } (x,t) \in \overline{\Omega} \times [t^\varepsilon, T]\,,\ 0 < \varepsilon << 1\,. \tag{17}$$

3.2. **Eternal solutions and planar waves.** We recall that a solution of an evolution equation is called *eternal* (or an *entire* solution) if it is defined for all positive and negative time. We follow this terminology to refer to a solution $w(z,\tau)$ of

$$w_\tau = \Delta_z w + f(w)\,, \quad z \in \mathbb{R}^N,\ \tau \in \mathbb{R}\,. \tag{18}$$

Stationary solutions and travelling waves are examples of eternal solutions. Crucial to our analysis is a recent result of Berestycki and Hamel [9] asserting that "any planar-like eternal solution is actually a planar wave". More precisely, the following holds (for $z \in \mathbb{R}^N$ we write $z = (z^{(1)}, \cdots, z^{(N)})$).

Lemma 3.2 ([9, Theorem 3.1])**.** *Let $w(z,\tau)$ be an eternal bounded solution of (18) satisfying*

$$\liminf_{z^{(N)}\to\infty} \inf_{z'\in\mathbb{R}^{N-1},\,\tau\in\mathbb{R}} w(z,\tau) > a\,, \quad \limsup_{z^{(N)}\to-\infty} \sup_{z'\in\mathbb{R}^{N-1},\,\tau\in\mathbb{R}} w(z,\tau) < a\,, \tag{19}$$

where $z' := (z^{(1)}, \cdots, z^{(N-1)})$. Then there exists a constant $z^* \in \mathbb{R}$ such that

$$w(z,\tau) = U_0(z^{(N)} - z^*)\,, \quad z \in \mathbb{R}^N\,,\ \tau \in \mathbb{R}\,.$$

3.3. **Proof of (ii) in Theorem 2.1.** In what follows we fix $\mu > 1$ and an arbitrary constant T_1 with $T < T_1 < T^{max}$ (see Remark 2). Obviously the conclusion of Lemma 3.1 remains valid if T is replaced by T_1. Assume by contradiction that (13) does not hold. Then there is $\eta > 0$ and sequences $\varepsilon_k \downarrow 0$, $t_k \in [\mu t^{\varepsilon_k}, T]$, $x_k \in \bar\Omega$ ($k = 1, 2, \ldots$) such that

$$\left| u^{\varepsilon_k}(x_k, t_k) - U_0\left(\frac{d^{\varepsilon_k}(x_k,t_k)}{\varepsilon_k}\right)\right| \ge 2\eta\,. \tag{20}$$

In view of (16)–(17) and $U_0(\pm\infty) = \alpha_\pm$, for (20) to hold it is necessary to have

$$d(x_k, t_k) = \mathcal{O}(\varepsilon_k)\,, \quad \text{as } k \to \infty\,. \tag{21}$$

If $u^{\varepsilon_k}(x_k,t_k) = a$, then this would mean that $x_k \in \Gamma_{t_k}^{\varepsilon_k}$, in which case the left-hand side of (20) would be 0 (since $U_0(0) = a$), which is impossible. Hence $u^{\varepsilon_k}(x_k,t_k) \ne a$. By extracting a subsequence if necessary, we may assume without loss of generality that $u^{\varepsilon_k}(x_k,t_k)-a$ has a constant sign for $k=0,1,2,\ldots$. Since the sign of this quantity is irrelevant in the later argument, in what follows we assume that

$$u^{\varepsilon_k}(x_k,t_k)>a \quad (k=0,1,2,\ldots), \tag{22}$$

which then implies that

$$d^{\varepsilon_k}(x_k,t_k)>0 \quad (k=0,1,2,\ldots).$$

Since the curvature of Γ_t is uniformly bounded for $0\le t\le T$, there is a $\delta>0$ such that each x in a δ-tubular neighborhood of Γ_t has a unique orthogonal projection on Γ_t. Since the sequence (x_k) remains very close to Γ_{t_k} by (21), each x_k (with sufficiently large k) has a unique orthogonal projection $p(x_k,t_k)\in\Gamma_{t_k}$. Let y_k be a point on $\Gamma_{t_k}^{\varepsilon_k}$ that has the smallest distance from x_k. If such a point is not unique, we choose one such point arbitrarily. Then we have

$$u^{\varepsilon_k}(y_k,t_k)=a \quad (k=0,1,2,\ldots), \tag{23}$$

$$d^{\varepsilon_k}(x_k,t_k)=\|x_k-y_k\|\,, \tag{24}$$

$$u^{\varepsilon_k}(x,t_k)>a \quad \text{if} \ \ \|x-x_k\|<\|y_k-x_k\|\,, \tag{25}$$

$$x_k-p_k\perp\Gamma_{t_k} \quad \text{at} \ \ p_k\in\Gamma_{t_k}\,,$$

where $p_k:=p(x_k,t_k)$. Furthermore, (21) and (17) imply

$$\|x_k-p_k\|=\mathcal{O}(\varepsilon_k)\,,\quad \|y_k-p_k\|=\mathcal{O}(\varepsilon_k)\quad (k=0,1,2,\ldots). \tag{26}$$

We now rescale the solution u^ε around (p_k,t_k) and define

$$w^k(z,\tau):=u^{\varepsilon_k}(p_k+\varepsilon_k\mathcal{R}_kz,t_k+\varepsilon_k^2\tau)\,, \tag{27}$$

where $\mathcal{R}_k$ is a matrix in $SO(N,\mathbb{R})$ that rotates the $z^{(N)}$ axis onto the normal at $p_k\in\Gamma_{t_k}$, that is,

$$\mathcal{R}_k:(0,\ldots,0,1)^T\mapsto n(p_k,t_k)\,,$$

where $(\ )^T$ denotes a transposed vector and $n(p,t)$ the outward normal unit vector at $p\in\Gamma_t$. Since Γ_t (hence the points p_k) is uniformly separated from $\partial\Omega$ by some positive distance, there exists $c>0$ such that w^k is defined (at least) on the box

$$B^k:=\left\{(z,\tau)\in\mathbb{R}^N\times\mathbb{R}\,:\,\|z\|\le\frac{c}{\varepsilon_k},\quad -(\mu-1)f'(a)^{-1}|\ln\varepsilon_k|\le\tau\le\frac{T_1-T}{\varepsilon_k^2}\right\}.$$

Since u^ε satisfies (P^ε), we see that w^k satisfies

$$w^k_\tau=\Delta_zw^k+f(w^k)-\varepsilon_kg^{\varepsilon_k}(p_k+\varepsilon_k\mathcal{R}_kz,t_k+\varepsilon_k^2\tau,w^k)\quad \text{in } B^k\,. \tag{28}$$

Moreover, if $(z,\tau)\in B^k$ then $t^{\varepsilon_k}\le t_k+\varepsilon_k^2\tau\le T_1$. Therefore (16) implies

$$\begin{cases} d(p_k+\varepsilon_k\mathcal{R}_kz,t_k+\varepsilon_k^2\tau)\le -C_0\varepsilon_k & \Rightarrow \quad w^k(z,\tau)\le\alpha_-+\eta\,,\\ d(p_k+\varepsilon_k\mathcal{R}_kz,t_k+\varepsilon_k^2\tau)\ge C_0\varepsilon_k & \Rightarrow \quad w^k(z,\tau)\ge\alpha_+-\eta\,,\end{cases} \tag{29}$$

so long as $(z,\tau)\in B^k$. Now we recall that the rotation by $\mathcal{R}_k$ of the $z^{(N)}$ axis is normal to Γ_{t_k} at $p_k:=p(x_k,t_k)$, and that the curvature of Γ_t is uniformly bounded for $0\le t\le T$. In view of this, and considering that $d(x,t)$ is uniformly Lipschitz continuous in t because of the boundedness of the normal speed of Γ_t, we see from (29) that there exists a constant $C>0$, which is independent of k, such that

$$z^{(N)}\le -C\Rightarrow w^k(z,\tau)\le\alpha_-+\eta\,,\quad z^{(N)}\ge C\Rightarrow w^k(z,\tau)\ge\alpha_+-\eta\,, \tag{30}$$

for all $(z,\tau)\in B^k$ with $\|z\|\le\sqrt{1/\varepsilon_k}$ and $|\tau|\le 1/\varepsilon_k$.

Now, since w^k solves (28), the uniform (w.r.t. $k \geq 0$) boundedness of w^k and standard parabolic estimates, along with the derivative bounds on g^ε, imply that w^k is uniformly bounded in $C^{2+\gamma,1+\frac{\gamma}{2}}_{loc}(B^1)$. We can therefore extract from (w^k) a subsequence that converges to some w in $C^{2,1}_{loc}(B^1)$. By repeating this on all B^k, we can find a subsequence of (w^k) that converges to some w in $C^{2,1}_{loc}(\mathbb{R}^N \times \mathbb{R})$ (note that $\cup_{k\geq 0} B^k = \mathbb{R}^N \times \mathbb{R}$). Passing to the limit in (28) yields

$$w_\tau = \Delta_z w + f(w) \quad \text{on } \mathbb{R}^N \times \mathbb{R}\,.$$

Hence we have constructed an eternal solution $w(z,\tau)$ which — in view of (30)— satisfies (19). Lemma 3.2 then implies that

$$w(z,\tau) = U_0(z^{(N)} - z^*) \tag{31}$$

for some $z^* \in \mathbb{R}$.

Now we define sequences of points (z_k), $(\tilde{z}_k)$ by

$$z_k := \frac{1}{\varepsilon_k}\mathcal{R}_k^{-1}(x_k - p_k)\,, \quad \tilde{z}_k := \frac{1}{\varepsilon_k}\mathcal{R}_k^{-1}(y_k - p_k)\,.$$

By (26), these sequences are bounded, so we may assume without loss of generality that they converge:

$$z_k \to z_\infty\,, \quad \tilde{z}_k \to \tilde{z}_\infty\,, \quad \text{as} \quad k \to \infty\,.$$

By the definition of the z coordinates, z_∞ must lie on the $z^{(N)}$ axis, that is,

$$z_\infty = (0, \ldots, 0, z_\infty^{(N)})^T\,.$$

It follows from (23) and (25) that

$$w(\tilde{z}_\infty, 0) = a\,, \qquad w(z,0) \geq a \;\text{ if }\; \|z - z_\infty\| \leq \|\tilde{z}_\infty - z_\infty\|\,. \tag{32}$$

Note that by (31), the level set $w(z,0) = a$ coincides with the hyperplane $z^{(N)} = z^*$, and recall that ${U_0}' > 0$. Therefore, in view of (31) and (32), we have either $\tilde{z}_\infty = z_\infty$, or that the ball of radius $\|\tilde{z}_\infty - z_\infty\|$ centered at z_∞ is tangential to the hyperplane $z^{(N)} = z^*$ at $\tilde{z}_\infty$. This implies that $\tilde{z}_\infty$, as well as z_∞, must also lie on the $z^{(N)}$ axis. Therefore

$$\tilde{z}_\infty = (0, \ldots, 0, z^*)^T\,,$$

and the inequality $w(z_\infty, 0) \geq a$ implies that $z_\infty^{(N)} \geq z^*$. On the other hand (24) implies $d^{\varepsilon_k}(x_k, t_k)/\varepsilon_k = \|x_k - y_k\|/\varepsilon_k = \|z_k - \tilde{z}_k\| \to \|z_\infty - \tilde{z}_\infty\| = z_\infty^{(N)} - z^*$. The assumption (20) then yields

$$\begin{aligned} 0 &= \left| w(z_\infty, 0) - U_0(z_\infty^{(N)} - z^*) \right| \\ &= \left| \lim_{k\to\infty} u^{\varepsilon_k}(x_k, t_k) - U_0\Big(\lim_{k\to\infty} \frac{d^{\varepsilon_k}(x_k, t_k)}{\varepsilon_k} \Big) \right| \\ &\geq 2\eta\,. \end{aligned}$$

This contradiction proves statement (ii) of Theorem 15. ∎

3.4. **Proof of (i) and (iii) in Theorem 2.1.** The proof of (i) below uses an argument similar to the proof of Corollary 4.8 in [19]. Fix $\mu > 1$. For a given $\eta \in (0, \min(a - \alpha_-, \alpha_+ - a))$ define $\varepsilon_0 > 0$ and $C_0 > 0$ as in Lemma 3.1. Then we claim that

$$\liminf_{\varepsilon \to 0} \inf_{x \in \mathcal{N}_{C_0 \varepsilon}(\Gamma_t),\, \mu t^\varepsilon \leq t \leq T} \nabla u^\varepsilon(x,t) \cdot n(p(x,t),t) > 0\,, \tag{33}$$

where $n(p,t)$ denotes the outward unit normal vector at $p \in \Gamma_t$. Indeed, assume by contradiction that there exist sequences $\varepsilon_k \downarrow 0$, $t_k \in [\mu t^{\varepsilon_k}, T]$, $x_k \in \mathcal{N}_{C_0 \varepsilon_k}(\Gamma_{t_k})$ $(k = 1, 2, ...)$ such that

$$\nabla u^{\varepsilon_k}(x_k, t_k) \cdot n(p_k, t_k) \leq 0\,,$$

where $p_k = p(x_k, t_k)$. By rescaling around (p_k, t_k) and using arguments similar to those in the proof of (ii), one can find a point z_∞ with $|z_\infty^{(N)}| \leq C_0$ such that

$$U_0{}'(z_\infty^{(N)}) \leq 0\,,$$

which contradicts the fact that $U_0{}' > 0$ and establishes (33). Since, in view of Lemma 3.1, $\Gamma_t^\varepsilon \subset \mathcal{N}_{C_0\varepsilon}(\Gamma_t)$, the estimate (33) implies that $\nabla u^\varepsilon(x,t) \neq 0$ for all $x \in \Gamma_t^\varepsilon$; hence by the implicit function theorem, Γ_t^ε is a smooth hypersurface in a neighborhood of any point on it. The fact that Γ_t^ε can be expressed as a graph over Γ_t also follows from (33). This proves statement (i) of Theorem 2.1.

Finally, statement (iii) follows immediately from statements (i), (ii) and (17). This completes the proof of Theorem 2.1. □

3.5. **Proof of Theorem 2.2.** As shown in [2, Section 7], the behavior of u^ε in the system $(\mathrm{RD}^\varepsilon)$ can be treated as a special case of (P^ε), by regarding v^ε as a given function and using a contraction mapping theorem. Thus the conclusion of Theorem 2.2 follows directly from Theorem 2.1. □

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E-mail address: malfaro@math.univ-montp2.fr
E-mail address: matano@ms.u-tokyo.ac.jp