General fractal conservation laws arising from a model of detonations in gases

Matthieu Alfaro[1] **and Jérôme Droniou**[2],

I3M, Université de Montpellier 2,
CC051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France.

Abstract

We consider a model of cellular detonations in gases. It consists in conservation laws with a non-local pseudo-differential operator whose symbol is asymptotically $|\xi|^\lambda$, where $0 < \lambda \leq 2$; it can be decomposed as the $\lambda/2$ fractional power of the Laplacian plus a convolution term. After defining the notion of entropy solution, we prove the well-posedness in the L^∞ framework. In the case where $1 < \lambda \leq 2$ we also prove a regularising effect. In the appendix, we show that the assumptions made to perform the mathematical study are satisfied by the considered physical model of detonations (for which $\lambda = 1$).

Key Words: conservation law, Fourier integral operator, entropy solution, splitting method, Lévy operator. ([3])

1 Introduction

This paper is concerned with the fractal conservation law

$$\partial_t u(t,x) + \mathrm{div}(f(u))(t,x) + \mathcal{G}[u(t,\cdot)](x) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N\,, \tag{1.1}$$

supplemented with L^∞ initial data

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N\,. \tag{1.2}$$

Here $f : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz-continuous and $\mathcal{G}$ denotes the non-local operator defined through the Fourier transform by

$$\mathcal{F}\left(\mathcal{G}[u(t,\cdot)]\right)(\xi) = |\xi|^\lambda H(\xi)\mathcal{F}\left(u(t,\cdot)\right)(\xi)\,, \tag{1.3}$$

with $0 < \lambda \leq 2$ and $H : \mathbb{R}^N \to \mathbb{R}$.

[1] malfaro@math.univ-montp2.fr, corresponding author.
[2] droniou@math.univ-montp2.fr
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In the case where $H \equiv 1$ the non-local operator $\mathcal{G}$ reduces to a positive multiple g_λ of the fractional power $(-\Delta)^{\lambda/2}$ of order $\lambda/2$ of the Laplacian (Lévy operator), and (1.1) is well understood. More precisely, for $\lambda = 2$ it corresponds to the classical viscous conservation law (we have $\mathcal{G} \propto -\Delta$), which is well-posed and gives rise to a unique smooth solution. The case $\lambda < 2$ has first been studied in [5], in which local-in-time well-posedness was proved (in H^s Sobolev spaces, in particular) with some restrictions on f or λ. For $1 < \lambda < 2$, the global well-posedness in the L^∞ framework and the regularising effect of this fractal conservation law were then proved in [14]. If $0 < \lambda \leq 1$ the global well-posedness in the L^∞ framework is obtained in [1] thanks to an entropy formulation. Last, if $0 < \lambda < 1$ the non regularising effect is studied in [3]: discontinuities in the initial data may persist and — even for smooth initial data — shocks may develop. Other behaviours of this equation are also known, such as asymptotic properties (see [6, 7], [4]).

Nevertheless, the physical context indicates that the case of a non-constant frequency function H is quite relevant. Indeed in the context of pattern formation in detonation waves [10], [11], equation (1.1) arises with a pseudo-differential operator defined not by the symbol $|\xi|^\lambda$ but by a symbol $|\xi|^\lambda H(\xi)$ with $H(\xi) \to 1$ as $|\xi| \to \infty$ (see the physical context below for more details). This is the case we intend to consider in this paper; more precisely we assume that H satisfies the following property.

Assumption 1. $\Pi := \mathcal{F}^{-1}(|\cdot|^\lambda(H(\cdot) - 1)) \in L^1(\mathbb{R}^N)$.

Remark 1.1 *(Generalisations).* Let us precise that a few relaxations of Assumption 1 can be handled by our analysis: Π may "contain" Dirac masses (so that an additional linear reaction term in the equation can be treated) and may depend on the time variable. We refer to Section 7 for such generalisations.

Note that "$\mathcal{F}^{-1}(|\cdot|^\lambda(H(\cdot)-1)) \in L^1(\mathbb{R}^N)$" is implied by "$|\cdot|^\lambda(H(\cdot)-1) \in H^s(\mathbb{R}^N)$ for some $s > N/2$" or "$|\cdot|^\lambda(H(\cdot) - 1) \in W^{N+1,1}(\mathbb{R}^N)$" (see also Appendix A for less straightforward situations where a generalisation of Assumption 1 can hold).

Under the above assumption, equation (1.1) can be recast as

$$\partial_t u + \operatorname{div}(f(u)) + g_\lambda[u] + \Pi * u = 0 \quad \text{on } (0,\infty) \times \mathbb{R}^N . \tag{1.4}$$

Our aim is to prove, for $0 < \lambda \leq 2$, the well-posedness of (1.4) in the L^∞ framework and, in the case $\lambda > 1$, a regularising effect.

The physical context

In the framework of overdriven detonations in gases in 2D, under proper physical assumptions and simplifications (see [10], [11]), the shock wave can be represented by an equation $\zeta = \beta(\tau, \eta)$; here, τ is the (renormalised) time, ζ and η are the longitudinal and transverse coordinates to the shock (more precisely, transformations of these coordinates taking into account the density of the gases), and β evolves following, at the zeroth-order (with respect to a small physical parameter), a linear wave equation.

Performing a formal expansion of β with respect to this small physical parameter, it can be shown that its first-order term β_1 satisfies, up to a normalisation of constants, the equation

$$\frac{\partial \beta_1}{\partial \tau} + \frac{1}{2}\left(\frac{\partial \beta_1}{\partial \eta}\right)^2 + \mathcal{G}[\beta_1] = 0\,. \tag{1.5}$$

In this circumstance, one information of interest is the creation and evolution of cusps, abrupt changes in $u := \frac{\partial \beta_1}{\partial \eta}$. From (1.5) one sees that u precisely follows (1.1) (with $t = \tau$, $N = 1$, $f(u) = \frac{1}{2}u^2$ and $x = \eta$). The operator $\mathcal{G}$ involved here is described, after re-normalisation, by (1.3) with $\lambda = 1$ and $H(\xi) = \sqrt{1 + W(i|\xi|)}$, where W, defined on the imaginary axis, is regular and satisfies $W(is) \sim b/s$ as $s \to \infty$ (with b constant).

Thanks to this property, we prove in the appendix that H satisfies the following assumption (with $\lambda = 1$).

Assumption 2. *There exists $c \in \mathbb{R}$ such that $\Pi := \mathcal{F}^{-1}(|\cdot|^\lambda(H(\cdot) - 1)) \in c\delta_0 + L^1(\mathbb{R}^N)$, with δ_0 the Dirac mass at* 0.

This assumption is a generalisation of Assumption 1 (which corresponds to the case $c = 0$), and consists in adding a linear reaction term cu to (1.4). In order to simplify the presentation we shall make the whole study under Assumption 1 and explain in Section 7 how to handle the more general Assumption 2. Hence our analysis covers the considered physical model.

2 Main results

Let us first recall that, for $0 < \lambda < 2$, the fractional Laplacian g_λ has the following integral representation (see e.g. [15]), valid for all $r > 0$ and all $\varphi \in C_c^\infty(\mathbb{R}^N)$:

$$\begin{aligned} g_\lambda[\varphi](x) \quad = \quad & -c_N(\lambda) \int_{|z|\geq r} \frac{\varphi(x+z) - \varphi(x)}{|z|^{N+\lambda}}\, dz \\ & -c_N(\lambda) \int_{|z|\leq r} \frac{\varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\, dz\,, \end{aligned} \tag{2.1}$$

where $c_N(\lambda)$ is a (known) positive constant. From this representation, [1] defines a notion of entropy solution to $\partial_t u + \mathrm{div}(f(u)) + g_\lambda[u] = 0$ with initial data $u_0 \in L^\infty(\mathbb{R}^N)$: for all $r > 0$, all entropy pair (η, Φ) and all non-negative

$\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$,

$$\int_0^\infty \int_{\mathbb{R}^N} (\eta(u)\partial_t\varphi + \Phi(u)\cdot\nabla\varphi) + \int_0^\infty G_{\lambda,r}[u,\eta,\varphi](t)dt + \int_{\mathbb{R}^N} \eta(u_0)\varphi(0,\cdot) \geq 0\,, \tag{2.2}$$

where, here and in the following,

$$G_{\lambda,r}[u,\eta,\varphi](t) := c_N(\lambda)\int_{\mathbb{R}^N}\int_{|z|\geq r} \eta'(u(t,x))\frac{u(t,x+z)-u(t,x)}{|z|^{N+\lambda}}\varphi(t,x)\,dzdx + c_N(\lambda)\int_{\mathbb{R}^N}\int_{|z|\leq r} \eta(u(t,x))\frac{\varphi(t,x+z)-\varphi(t,x)-\nabla\varphi(t,x)\cdot z}{|z|^{N+\lambda}}\,dzdx\,.$$

This notion of entropy solution ensures the well-posedness in the L^∞ framework of the equation $\partial_t u + \mathrm{div}(f(u)) + g_\lambda[u] = 0$.

If $\lambda = 2$, $g_2[u] = -c_N(2)\Delta u$ and the definition of $G_{\lambda,r}$ must naturally be changed into

$$G_{2,r}[u,\eta,\varphi](t) := c_N(2)\int_{\mathbb{R}^N} \eta(u)\Delta\varphi\,.$$

Our definition of entropy solution to ((1.4),(1.2)) is a straightforward extension of this definition from [1].

Definition 2.1 (Entropy solution). An entropy solution to (1.4) with initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ is a function u belonging to $L^\infty((0,T)\times\mathbb{R}^N)$ for all $T>0$ and such that, for all $r>0$, all non-negative $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^N)$, all convex function $\eta \in C^1(\mathbb{R})$ and all function $\Phi:\mathbb{R}\to\mathbb{R}^N$ such that $\nabla\Phi = \eta'\nabla f$, we have

$$\int_0^\infty \int_{\mathbb{R}^N} (\eta(u)\partial_t\varphi + \Phi(u)\cdot\nabla\varphi) + \int_0^\infty G_{\lambda,r}[u,\eta,\varphi](t)\,dt - \int_0^\infty\int_{\mathbb{R}^N} \eta'(u)\varphi\,(\Pi * u) + \int_{\mathbb{R}^N} \eta(u_0)\varphi(0,\cdot) \geq 0\,. \tag{2.3}$$

Remark 2.2. Note that, as in the case of pure conservation laws, one can replace the smooth pairs (η,Φ) in this definition by Kruzhkov's entropy pairs [16] without changing the notion of entropy solution. For a given Kruzhkov entropy $\eta(s) = |s-\kappa|$, the value of η' at $s=\kappa$ to be considered in (2.3) can be any element of the sub-differential $[-1,1]$ of η at $s=\kappa$.

Thanks to this definition, we will prove the well-posedness of the considered equation.

Theorem 2.3 (Well-posedness). *Let $0<\lambda\leq 2$ and $u_0 \in L^\infty(\mathbb{R}^N)$. Let Assumption 1 be satisfied. Then there exists a unique entropy solution u to ((1.4),(1.2)). Moreover, u is continuous $[0,\infty)\to L^1_{\mathrm{loc}}(\mathbb{R}^N)$.*

Remark 2.4. Note that our analysis also covers the elementary situation $\lambda = 0$, in which case $g_0[u] = u$ and $G_{0,r}[u,\eta,\varphi] = -\int_{\mathbb{R}^N} \eta'(u)u\varphi$.

Remark 2.5. The use of an entropy formulation is mandatory. Indeed, it has been proved in [2] that, even for the simplest case where $\Pi = 0$, the notion of weak solution is not strong enough to provide uniqueness if $\lambda < 1$.

We will also obtain, for $\lambda > 1$, a regularising effect.

Theorem 2.6 (Regularising effect). *Let $1 < \lambda \leq 2$ and $u_0 \in L^\infty(\mathbb{R}^N)$. Let Assumption 1 be satisfied. Then the entropy solution u to ((1.4),(1.2)) is smooth for $t > 0$; more precisely, for all $0 < a < T$, $u \in C_b^\infty((a,T) \times \mathbb{R}^N)$.*

Remark 2.7. As mentioned in the introduction, it is known that for $\lambda < 1$ the regularising effect does not occur. In fact, in this case, shocks can occur [9] even with smooth initial data [3], although these shocks can sometimes disappear if $\Pi = 0$ (i.e. $\mathcal{G} = g_\lambda$), the initial data belongs to L^2 and the exponent λ is not too far from 1.

For $\lambda = 1$ and $f(u) = u^2$, it is proved in [8] that if $\Pi = 0$ and if the initial data belongs to L^2 then the regularising effect occurs. However, the situation with a merely bounded initial data or with $\Pi \neq 0$ is not clear, the techniques in [8] being strongly based on a scaling that is only true for the pure fractal Burgers equation. In particular, for the physical context described in the introduction (which corresponds to $\lambda = 1$ and $\Pi \neq 0$), the regularity or loss of regularity is still an open question.

The organisation of the paper is as follows. In Section 3 we introduce notations and useful preliminary results. By using a splitting method we construct an entropy solution in Section 4. Uniqueness of the solution is proved via a "finite speed propagation property" in Section 5. In Section 6, by taking advantage of a Duhamel's formula for $1 < \lambda \leq 2$ we prove Theorem 2.6. A few generalisations are discussed in Section 7. Last, the consistency with the physical context is proved in Appendix A.

3 Notations and preliminary remarks

Before proving our results, we introduce some notations. Let

$$K(t) := \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda}).$$

The (unique bounded) solution to $\partial_t u + g_\lambda[u] = 0$ with initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ is given by $u(t) = K(t) * u_0$.

For any integrable function α, we define

$$S_{-\alpha}(t) := \delta_0 + \sum_{n\geq 1} \frac{t^n}{n!}(-\alpha)^{*(n)},$$

where δ_0 is the Dirac mass at 0 and $(-\alpha)^{*(n)} := (-\alpha) * \cdots * (-\alpha)$ is the convolution of $-\alpha$ with itself $n-1$ times. The (unique) bounded solution to $\partial_t u + \alpha * u = 0$ with initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ is given by $u(t) = S_{-\alpha}(t) * u_0$ ([4]).

In several proofs to come, we denote

$$K^{[2]}(t) := K(2t) \quad \text{and} \quad {S_{-\alpha}}^{[2]}(t) := S_{-\alpha}(2t)\,,$$

namely the semi-groups associated with $\partial_t u + 2\, g_\lambda[u] = 0$ and $\partial_t u + 2\,\alpha * u = 0$.

Let us state the main properties of K and $S_{-\alpha}$.

Proposition 3.1 (Properties of the kernels). *For all $0 < \lambda \le 2$ and all $\alpha \in L^1(\mathbb{R}^N)$, the kernels K and $S_{-\alpha}$ satisfy the following properties.*

(i) *K is positive and, for all $t > 0$, $K(t) \in L^1(\mathbb{R}^N)$, $||K(t)||_{L^1(\mathbb{R}^N)} = 1$ and, for all $x \in \mathbb{R}^N$, $K(t,x) = t^{-N/\lambda} K(1, t^{-1/\lambda} x)$.*

(ii) *$K \in C_b^\infty((a,\infty) \times \mathbb{R}^N)$ for all $a > 0$, and there exists $C > 0$ such that, for all $t > 0$, $||\nabla K(t)||_{L^1(\mathbb{R}^N)} \le C t^{-1/\lambda}$.*

(iii) *For all $t,\ s > 0$, $K(t) * K(s) = K(t+s)$ and $(\nabla K(t)) * K(s) = \nabla K(t+s)$.*

(iv) *The functions $t \in (0,\infty) \mapsto K(t) \in L^1(\mathbb{R}^N)$ and $t \in (0,\infty) \mapsto \nabla K(t) \in L^1(\mathbb{R}^N)^N$ are continuous.*

(v) *For all $t,\ s > 0$, $S_{-\alpha}(t) * S_{-\alpha}(s) = S_{-\alpha}(t+s)$.*

(vi) *The function $t \in [0,\infty) \mapsto S_{-\alpha}(t) - \delta_0 \in L^1(\mathbb{R}^N)$ is continuous.*

(vii) *For all $t > 0$, the functions $K(t) * S_{-\alpha}(t)$ and $\nabla K(t) * S_{-\alpha}(t)$ belong to $C_b^\infty(\mathbb{R}^N)$.*

(viii) *The functions $(t,s) \in (0,\infty)^2 \mapsto K(t) * S_{-\alpha}(s) \in L^1(\mathbb{R}^N)$ and $(t,s) \in (0,\infty)^2 \mapsto \nabla K(t) * S_{-\alpha}(s) \in L^1(\mathbb{R}^N)^N$ are continuous. Moreover, there exists $C > 0$ such that, for all $t,\ s > 0$, $||K(t) * S_{-\alpha}(s)||_{L^1(\mathbb{R}^N)} \le C e^{||\alpha||_1 s}$ and $||\nabla K(t) * S_{-\alpha}(s)||_{L^1(\mathbb{R}^N)} \le C e^{||\alpha||_1 s} t^{-1/\lambda}$.*

Proof. The properties on K are quite classical and, aside from its positivity, can be deduced straightforwardly from its definition (see also [14], [15]); the positivity of K can be found in [17], [14].

Property (v) is the expression of the fact that $S_{-\alpha}$ is a semi-group (in fact, a group...), and property (vi) is a consequence of the normal convergence, in $C([0,T]; L^1(\mathbb{R}^N))$, of the series $S_{-\alpha}(t) - \delta_0 = \sum_{n \ge 1} \frac{t^n}{n!} (-\alpha)^{*(n)}$. Finally, properties (vii) and (viii) come from the writing $X * S_{-\alpha}(s) =$

[4]Obviously, though the convolution of a Dirac mass by an L^∞ function is not pointwise well defined, we let $\delta_0 * u_0 = u_0$.

$X + X * (S_{-\alpha}(t) - \delta_0)$ (with $X = K(t)$ or $X = \nabla K(t)$), from items (ii), (iv), (vi) and from the estimate $||S_{-\alpha}(s) - \delta_0||_{L^1(\mathbb{R}^N)} \le \sum_{s\ge 1} \frac{s^n}{n!} ||\alpha||_1^n \le e^{||\alpha||_1 s}$. ∎

We will also need the following estimate on g_λ.

Lemma 3.2. *Let $\lambda \in (0, 2]$. There exists $C_\lambda > 0$ such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,*

$$\|g_\lambda[\varphi]\|_{L^1(\mathbb{R}^N)} \le C_\lambda \|\varphi\|_{W^{2,1}(\mathbb{R}^N)} .$$

In particular, g_λ can be extended into a linear continuous operator from $W^{2,1}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$.

Proof. The property for $\lambda = 2$ is obvious (since, up to a multiplicative constant, g_λ is the Laplace operator). We thus consider that $\lambda < 2$ and we use the integral representation (2.1) of g_λ with $r = 1$ and a Taylor expansion to write $|g_\lambda[\varphi](x)| \le T_1[\varphi](x) + T_2[\varphi](x)$ with

$$T_1[\varphi](x) = c_N(\lambda) \int_{|z|\ge 1} \frac{|\varphi(x+z)| + |\varphi(x)|}{|z|^{N+\lambda}}\, dz \,,$$

and

$$T_2[\varphi](x) = c_N(\lambda) \int_{|z|\le 1} \frac{\int_0^1 \frac{1}{2} |D^2\varphi(x+sz)|\, |z|^2\, ds}{|z|^{N+\lambda}}\, dz \,,$$

where $|D^2\varphi|$ is the Euclidean matrix norm of $D^2\varphi$. Then, using Fubini-Tonelli's theorem and linear changes of variable, we find

$$\begin{aligned} \int_{\mathbb{R}^N} T_1[\varphi](x)\, dx &= c_N(\lambda) \int_{|z|\ge 1} \frac{\int_{\mathbb{R}^N} |\varphi(x+z)|\, dx + \int_{\mathbb{R}^N} |\varphi(x)|\, dx}{|z|^{N+\lambda}}\, dz \\ &= 2 c_N(\lambda) \|\varphi\|_{L^1(\mathbb{R}^N)} \int_{|z|\ge 1} \frac{dz}{|z|^{N+\lambda}} \,, \end{aligned}$$

with $N + \lambda > N$, and

$$\begin{aligned} \int_{\mathbb{R}^N} T_2[\varphi](x)\, dx &= c_N(\lambda) \int_{|z|\le 1} \frac{\int_0^1 \frac{1}{2} \left(\int_{\mathbb{R}^N} |D^2\varphi(x+sz)|\, dx \right) ds}{|z|^{N+\lambda-2}}\, dz \\ &= \frac{c_N(\lambda)}{2} \|\, |D^2\varphi|\, \|_{L^1(\mathbb{R}^N)} \int_{|z|\le 1} \frac{dz}{|z|^{N+\lambda-2}} \,, \end{aligned}$$

with $N + \lambda - 2 < N$. The proof is complete. ∎

4 Existence of an entropy solution

By using the splitting method developed in [14] and later in [1] we construct an entropy solution to ((1.4),(1.2)).

For $\delta > 0$ we define $u^\delta : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ as follows. Let $u^\delta(0, \cdot) := u_0$ and, for all $n \ge 0$, define by induction

- u^δ on $(2n\delta, (2n+1)\delta] \times \mathbb{R}^N$ as the (entropy) solution to

$$\partial_t u + 2\,\mathrm{div}(f(u)) + 2\,g_\lambda[u] = 0\,, \tag{4.1}$$

  supplemented with the initial data $u^\delta(2n\delta, \cdot)$.

- u^δ on $((2n+1)\delta, (2n+2)\delta] \times \mathbb{R}^N$ as the (unique bounded) solution to

$$\partial_t u + 2\,\Pi * u = 0\,, \tag{4.2}$$

  supplemented with the initial data $u^\delta((2n+1)\delta, \cdot)$.

Note that equation (4.1) does not increase the L^∞ norm and that its solutions are continuous with values in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ (see [1] for instance). On the other hand, the representation $u(t) = S_{-2\Pi}(t-s) * u(s)$ of the solutions to (4.2) show that they satisfy $\|u(t)\|_\infty \leq e^{2\|\Pi\|_1(t-s)}\|u(s)\|_\infty$ for $t \geq s$, and also that they are continuous with values in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. In particular, at each step the functions $u^\delta(2n\delta, \cdot)$ and $u^\delta((2n+1)\delta, \cdot)$ are bounded and thus suitable initial data for the considered equations.

Therefore we are equipped with $u^\delta \in C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^N))$ such that

$$\|u^\delta(t)\|_\infty \leq e^{\|\Pi\|_1 t}\|u_0\|_\infty\,. \tag{4.3}$$

By Arzéla-Ascoli's theorem, we first prove the relative compactness of $\{u^\delta : 0 < \delta < T\}$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. Then by extraction of a subsequence as $\delta \to 0$ we construct an entropy solution to ((1.4),(1.2)).

4.1 Relative compactness in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$

Step 1. We fix $T \geq 0$ and prove that $\{u^\delta(t) : 0 < \delta < T\,,\, t \in [0,T]\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

For a given u we define $\mathcal{T}_h u$ the associated translated function of u by $\mathcal{T}_h u(t,x) := u(t, x+h)$. Note that $\mathcal{T}_h u^\delta$ solves (4.1) and (4.2) on the intervals where u^δ solves these equations.

We recall that the kernel associated to equation $\partial_t u + 2\,g_\lambda[u] = 0$ is nothing else but $K(2t) =: K^{[2]}(t)$, and quote [1, Theorem 3.2] — which can be seen as a *finite speed propagation* property for equation (4.1):

Lemma 4.1. *Let u and v be the entropy solutions to* (4.1) *with initial conditions u_0 and v_0 in L^∞. Then, for all $x_0 \in \mathbb{R}^N$, all $t > 0$, all $R > 0$,*

$$\int_{B(x_0,R)} |u - v|(t) \leq \int_{B(x_0,R+2Lt)} K^{[2]}(t) * |u_0 - v_0|\,,$$

where L is a Lipschitz constant of f on $\{s \in \mathbb{R} : |s| \leq \max(\|u_0\|_\infty, \|v_0\|_\infty)\}$ and $B(x_0,R)$ is the ball in $\mathbb{R}^N$ of center x_0 and radius R.

In view of (4.3), by selecting L as a Lipschitz constant of f on the interval $[-e^{\|\Pi\|_1 T}\|u_0\|_\infty, e^{\|\Pi\|_1 T}\|u_0\|_\infty]$, we can apply the above lemma, with $(u,v) = (u^\delta, \mathcal{T}_h u^\delta)$, on all intervals of $[0,T]$ where u^δ (and so $\mathcal{T}_h u^\delta$) solves (4.1).

Let $t \in [0,T]$. Assume that $2n\delta < t \leq (2n+1)\delta$, for some $n \geq 0$. Then it follows from Lemma 4.1 that, denoting $B(R) = B(0,R)$,

$$\begin{aligned}\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) &\leq \int_{B(R+2L(t-2n\delta))} K^{[2]}(t-2n\delta) * |u^\delta - \mathcal{T}_h u^\delta|(2n\delta) \\ &\leq \int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - \mathcal{T}_h u^\delta|(2n\delta)\,, \end{aligned} \tag{4.4}$$

thanks to the positivity of the kernel K. Now, if $n \neq 0$ we go further in the past. Since

$$\partial_t(u^\delta - \mathcal{T}_h u^\delta) + 2\,(\Pi - \mathcal{T}_h \Pi) * u^\delta = 0 \qquad \text{on } ((2n-1)\delta, 2n\delta]\,,$$

we have, on the above time interval,

$$\begin{aligned}\|\partial_t(u^\delta - \mathcal{T}_h u^\delta)(t)\|_\infty &\leq 2\|\Pi - \mathcal{T}_h\Pi\|_1 \|u^\delta(t)\|_\infty \\ &\leq 2\|\Pi - \mathcal{T}_h\Pi\|_1 e^{\|\Pi\|_1 T}\|u_0\|_\infty =: \omega_T(h)\,,\end{aligned}$$

with $\omega_T(h)$ not depending on δ and $\omega_T(h) \to 0$ as $h \to 0$. It follows that, for all $x \in \mathbb{R}^N$,

$$|u^\delta - \mathcal{T}_h u^\delta|(2n\delta, x) \leq \omega_T(h)\delta + |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta, x)\,. \tag{4.5}$$

By plugging this into (4.4), using $||K(t)||_1 = 1$ and $B(R+2L\delta) \subset B(R+2LT)$, we find that

$$\begin{aligned}&\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \\ &\leq \int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta) + \omega_T(h)\delta|B(R+2LT)|\,.\end{aligned} \tag{4.6}$$

In order to estimate the first term in the right hand side member we notice that u^δ and $\mathcal{T}_h u^\delta$ solve (4.1) on $((2n-2)\delta, (2n-1)\delta]$ and thus, applying

Lemma 4.1, we find:

$$
\begin{aligned}
&\int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta) \\
&= \int_{\mathbb{R}^N} K^{[2]}(t-2n\delta, y) \int_{B(R+2L\delta)} |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta, x-y)\, dx\, dy \\
&\le \int_{\mathbb{R}^N} K^{[2]}(t-2n\delta, y) \\
&\qquad\qquad \int_{B(R+4L\delta)} \left[K^{[2]}(\delta,\cdot) * |u^\delta - \mathcal{T}_h u^\delta|((2n-2)\delta,\cdot)\right](x-y)\, dx\, dy \\
&\le \int_{B(R+4L\delta)} \Big\{K^{[2]}(t-2n\delta,\cdot) * \\
&\qquad\qquad\qquad \left[K^{[2]}(\delta,\cdot) * |u^\delta - \mathcal{T}_h u^\delta|((2n-2)\delta,\cdot)\right]\Big\}(x)\, dx \\
&\le \int_{B(R+4L\delta)} K^{[2]}(t-(2n-1)\delta) * |u^\delta - \mathcal{T}_h u^\delta|((2n-2)\delta)\,.
\end{aligned}
$$

We plug this into (4.6) to get

$$
\begin{aligned}
&\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \\
&\qquad \le \int_{B(R+4L\delta)} K^{[2]}(t-(2n-1)\delta) * |u^\delta - \mathcal{T}_h u^\delta|((2n-2)\delta) \\
&\qquad\qquad\qquad\qquad + \omega_T(h)\delta|B(R+2LT)|\,. \qquad (4.7)
\end{aligned}
$$

By repeating $n-1$ more times the procedure from (4.5) to (4.7), we discover that

$$
\begin{aligned}
&\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \\
&\le \int_{B(R+2L(n+1)\delta)} K^{[2]}(t-n\delta) * |u_0 - \mathcal{T}_h u_0| + \omega_T(h) n\delta |B(R+2LT)| \\
&\le \sup_{0\le s\le T} \int_{B(R+2LT)} K^{[2]}(s) * |u_0 - \mathcal{T}_h u_0| + \omega_T(h) T |B(R+2LT)|\,, \quad (4.8)
\end{aligned}
$$

the last line following from $0 \le t - n\delta \le (n+1)\delta \le 2n\delta \le t \le T$.

Assume that $(2n+1)\delta < t \le (2n+2)\delta$, for some $n \ge 0$. By using similar arguments, we claim that we obtain (4.8) again.

Applying [1, Lemma A.2] with $\varepsilon = 1$, we deduce from (4.8) that

$$
\begin{aligned}
&\sup_{0<\delta<T} \sup_{0\le t\le T} \int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \le \|u_0 - \mathcal{T}_h u_0\|_{L^1(B(R+2LT+r))} \\
&+2\|u_0\|_\infty |B(R+2LT)| \int_{\mathbb{R}^N\setminus B(r/T^{1/\lambda})} K^{[2]}(1) + \omega_T(h) T |B(R+2LT)|\,,
\end{aligned}
$$

holds for all $r > 0$. We conclude by a "3ε argument": if $\varepsilon > 0$ is given we fix $r > 1$ large enough so that $0 \leq \int_{\mathbb{R}^N \setminus B(r/T^{1/\lambda})} K^{[2]}(1) \leq \varepsilon$; since $u_0 \in L^\infty(\mathbb{R}^N) \subset L^1(B(R+2LT+r))$ we have $\|u_0 - \mathcal{T}_h u_0\|_{L^1(B(R+2LT+r))} \leq \varepsilon$ for h small enough; recall also that $\omega_T(h) \leq \varepsilon$ for h small enough. Therefore

$$\lim_{h\to 0} \sup_{0<\delta<T} \sup_{0\leq t\leq T} \int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) = 0\,,$$

which concludes the first step, by the Riesz-Fréchet-Kolmogorov's theorem.

Step 2. Still fixing $T > 0$, we prove that, for all Q compact subset of $\mathbb{R}^N$, $\{u^\delta : 0 < \delta < T\}$ is equicontinuous $[0,T] \to L^1(Q)$.

From (4.3), we see that $\{u^\delta(t) : 0 < \delta < T\,,\, t \in [0,T]\}$ is bounded in $L^\infty(\mathbb{R}^N)$. Since $\{u^\delta : 0 < \delta < T\}$ is bounded in $L^\infty((0,T)\times\mathbb{R}^N)$, in view of Lemma 3.2 we see ([5]) that $\{\Pi * u^\delta : 0 < \delta < T\}$ and $\{\mathrm{div}(f(u^\delta)) + g_\lambda[u^\delta] : 0 < \delta < T\}$ are bounded in $L^\infty(0,T;W^{-2,\infty}(\mathbb{R}^N))$, where we recall that $W^{-2,\infty}$ denotes the dual space of $W^{2,1}$.

Hence, equations (4.1) and (4.2), which are satisfied in the distributional sense, show that $\{\partial_t u^\delta : 0 < \delta < T\}$ is bounded in $L^\infty(0,T;W^{-2,\infty}(\mathbb{R}^N))$. We deduce that $\{u^\delta : 0 < \delta < T\,,\, t \in [0,T]\}$ is uniformly Lipschitz-continuous $[0,T] \to W^{-2,\infty}(\mathbb{R}^N)$, and thus also $[0,T] \to (C_c^2(Q))'$ (where $(C_c^2(Q))'$ is the dual space of $C_c^2(Q)$ endowed with the norm $||\varphi||_{C_c^2(Q)} = \sup_{|\alpha|\leq 2} ||\partial^\alpha\varphi||_\infty$).

We then need the following Lemma which can be considered as a metric-space variant of the classical Lions "three-spaces" lemma.

Lemma 4.2. *Let (E, d_E) and (F, d_F) be metric vector spaces such that E is continuously embedded in F; let $\mathcal{K}$ be a compact subset of E. Then, for all $\varepsilon > 0$, there exists $C_{\mathcal{K},\varepsilon} > 0$ such that, for all $(x,y) \in \mathcal{K}^2$, $d_E(x,y) \leq \varepsilon + C_{\mathcal{K},\varepsilon} d_F(x,y)$.*

Proof. The proof can be made by way of contradiction. Given $\varepsilon > 0$, if for all integer n we can find $(x_n, y_n) \in \mathcal{K}^2$ such that $d_E(x_n,y_n) > \varepsilon + n d_F(x_n,y_n)$, then — up to a subsequence — we can assume that $(x_n,y_n) \to (x,y)$ in E, and thus in F. Letting $n \to \infty$ in $d_F(x_n,y_n) < \frac{1}{n} d_E(x_n,y_n)$ we deduce that $d_F(x,y) = 0$ so that $x = y$. Letting then $n \to \infty$ in $\varepsilon < d_E(x_n,y_n)$ we see that $\varepsilon \leq 0$, which is a contradiction. This concludes the proof.■

Let us now conclude the proof that $\{u^\delta : 0 < \delta < T\}$ is equicontinuous $[0,T] \to L^1(Q)$. Let M be a uniform (independent on δ) Lipschitz constant of $u^\delta : [0,T] \to (C_c^2(Q))'$. If we denote by $\mathcal{K}$ the closure of $\{u^\delta(t) : 0 < \delta < T\,,\, t \in [0,T]\}$ in $L^1(Q)$, we have from Step 1 that $\mathcal{K}$ is compact in $L^1(Q)$. Let $\varepsilon > 0$ and select $C_{\mathcal{K},\varepsilon} > 0$ as in Lemma 4.2 applied to $E = L^1(Q)$ and

[5] It suffices to notice that, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have $|\langle \Pi * u^\delta(t), \varphi\rangle| \leq \|\Pi\|_1 \|u^\delta(t)\|_\infty \|\varphi\|_1$ and $|\langle \mathrm{div}(f(u^\delta(t))), \varphi\rangle| = |\langle f(u^\delta(t)), \nabla\varphi\rangle| \leq \|f(u^\delta(t))\|_\infty \|\nabla\varphi\|_1$ and $|\langle g_\lambda[u^\delta(t)], \varphi\rangle| = |\langle u^\delta(t), g_\lambda[\varphi]\rangle| \leq C\|u^\delta(t)\|_\infty \|\varphi\|_{W^{2,1}}$.

$F = (C_c^2(Q))'$. Then, if $(t,s) \in [0,T]^2$ are such that $|t-s| \leq \varepsilon/(MC_{\mathcal{K},\varepsilon})$, we have, for all $\delta > 0$,

$$d_{L^1(Q)}(u^\delta(t), u^\delta(s)) \leq \varepsilon + C_{\mathcal{K},\varepsilon}\, d_{(C_c^2(Q))'}(u^\delta(t), u^\delta(s)) \leq \varepsilon + C_{\mathcal{K},\varepsilon}\, M|t-s| \leq 2\varepsilon\,,$$

and the equicontinuity of $\{u^\delta \,:\, 0 < \delta < T\}$ on $[0,T]$ with values in $L^1(Q)$ is proved.

Conclusion. Gathering Steps 1 and 2, we conclude that $\{u^\delta \,:\, 0 < \delta < T\}$ is relatively compact in $C([0,T]; L^1_{\rm loc}(\mathbb{R}^N))$ for all $T > 0$.

4.2 Convergence to an entropy solution

Up to a subsequence, we can assume that, as $\delta \to 0$, u^δ converges to some u in $C([0,T]; L^1_{\rm loc}(\mathbb{R}^N))$ for all $T > 0$. Obviously, u also satisfies (4.3) and thus belongs to $L^\infty((0,T) \times \mathbb{R}^N)$ for all $T > 0$. We now prove that u is an entropy solution to (1.4) with initial data $u_0 \in L^\infty(\mathbb{R}^N)$.

Let $r > 0$, $\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$ be non-negative, $\eta \in C^1(\mathbb{R})$ be convex and $\Phi : \mathbb{R} \to \mathbb{R}^N$ be such that $\nabla\Phi = \eta'\nabla f$.

First, we claim that from (2.2) we can deduce an "entropy formulation with final value" for solutions to (4.1). More precisely, if v is the entropy solution to (4.1) with initial data v_0 then, for all $s > 0$,

$$\int_0^s \int_{\mathbb{R}^N} (\eta(v)\partial_t\varphi + 2\Phi(v)\cdot\nabla\varphi) + 2\int_0^s G_{\lambda,r}[v,\eta,\varphi](t)\,dt \\ + \int_{\mathbb{R}^N} \eta(v_0)\varphi(0,\cdot) - \int_{\mathbb{R}^N} \eta(v(s,\cdot))\varphi(s,\cdot) \geq 0\,. \tag{4.9}$$

Indeed, take $\gamma_\varepsilon : [0,\infty) \to [0,1]$ which tends to the characteristic function of $[0,s]$ as $\varepsilon \to 0$ and such that $-\gamma_\varepsilon'$ tends to the Dirac mass at $t = s$, and apply the entropy formulation (2.2) with $\varphi(t,x)$ replaced by $\varphi(t,x)\gamma_\varepsilon(t)$; letting $\varepsilon \to 0$, and since $v \in C([0,\infty); L^1_{\rm loc}(\mathbb{R}^N))$ — see [1] — we deduce that (4.9) holds.

The definition of u^δ then ensures that, for all $n \geq 0$,

$$\int_{2n\delta}^{(2n+1)\delta} \int_{\mathbb{R}^N} (\eta(u^\delta)\partial_t\varphi + 2\Phi(u^\delta)\cdot\nabla\varphi) + 2\int_{2n\delta}^{(2n+1)\delta} G_{\lambda,r}[u^\delta,\eta,\varphi](t)\,dt \\ + \int_{\mathbb{R}^N} \eta(u^\delta(2n\delta,\cdot))\varphi(2n\delta,\cdot) \\ - \int_{\mathbb{R}^N} \eta(u^\delta((2n+1)\delta,\cdot))\varphi((2n+1)\delta,\cdot) \geq 0\,. \tag{4.10}$$

On the other hand, multiplying (4.2) by $\eta'(u^\delta)\varphi$ and integrating by parts

([6]), we have, for all $n \geq 0$,

$$\int_{(2n+1)\delta}^{(2n+2)\delta} \int_{\mathbb{R}^N} \eta(u^\delta)\partial_t\varphi - 2\eta'(u^\delta)\varphi\,(\Pi * u^\delta)$$
$$+ \int_{\mathbb{R}^N} \eta(u^\delta((2n+1)\delta,\cdot))\varphi((2n+1)\delta,\cdot)$$
$$- \int_{\mathbb{R}^N} \eta(u^\delta((2n+2)\delta,\cdot))\varphi((2n+2)\delta,\cdot) = 0\,. \tag{4.11}$$

Summing (4.10) and (4.11) on all $n \geq 0$ (note that since φ is compactly supported, the sum is actually made of a finite number of terms), all the boundary terms but the first one cancel out each other and we find

$$\int_0^\infty \int_{\mathbb{R}^N} (\eta(u^\delta)\partial_t\varphi + 2I_\delta\,\Phi(u^\delta)\cdot\nabla\varphi) + \int_0^\infty 2I_\delta(t)G_{\lambda,r}[u^\delta,\eta,\varphi](t)\,dt$$
$$- \int_0^\infty 2J_\delta(t)\int_{\mathbb{R}^N} \eta'(u^\delta)\varphi\,\Pi * u^\delta + \int_{\mathbb{R}^N} \eta(u_0)\varphi(0,\cdot) \geq 0\,, \tag{4.12}$$

where I_δ is the characteristic function of $\cup_{n\geq 0}(2n\delta,(2n+1)\delta]$ and J_δ is the characteristic function of $\cup_{n\geq 0}((2n+1)\delta,(2n+2)\delta]$.

It is classical that, as $\delta \to 0$, both I_δ and J_δ tend to the constant function $1/2$ in $L^\infty(0,\infty)$ weak-$*$. Select $T > 0$ large enough so that $\operatorname{supp}\varphi \subset [0,T]\times\mathbb{R}^N$. We claim that the functions $t \mapsto \int_{\mathbb{R}^N} \Phi(u^\delta)\cdot\nabla\varphi$, $t \mapsto G_{\lambda,r}[u^\delta,\eta,\varphi](t)$ and $t \mapsto \int_{\mathbb{R}^N} \eta'(u^\delta)\varphi\,(\Pi * u^\delta)$ tend in $L^1(0,\infty)$ to the same quantities with u^δ replaced by u; indeed, let $A[u^\delta]$ be any one of these three functions: from $u^\delta \to u$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$, we deduce that $A[u^\delta](t) \to A[u](t)$ for $0 \leq t \leq T$, and from $\sup_{0<\delta<T}\sup_{0\leq t\leq T}|A[u^\delta](t)| < \infty$ and $A[u^\delta] \equiv 0$ on (T,∞), we infer that $A[u^\delta] \to A[u]$ in $L^1(0,\infty)$.

We can therefore pass to the limit $\delta \to 0$ in (4.12), to conclude that u satisfies (2.3) and is an entropy solution to (1.4) with initial condition u_0.

5 Uniqueness of the entropy solution

The uniqueness of the entropy solution will be obtained while proving the following "finite speed propagation" property.

Proposition 5.1 (Finite speed propagation). *Let u and v be entropy solutions to* (1.4) *with initial conditions u_0 and v_0 in L^∞ and let $T > 0$. Define*

$$m_0(T) := e^{\|\Pi\|_1 T}\max\{\|u_0\|_\infty, \|v_0\|_\infty\}.$$

[6]This is possible since $\partial_t u^\delta(\cdot,x) \in C([0,T],\mathbb{R})$. Indeed from $u^\delta \in C([0,T];L^1_{\mathrm{loc}})$ and $\sup_t \|u^\delta(t)\|_\infty < \infty$ we deduce that $u^\delta \in C([0,T];L^\infty_{\text{weak-}*})$. Combined with the continuity of $v \in L^\infty_{\text{weak-}*} \to \Pi * v(x) \in \mathbb{R}$ this shows that $\Pi * u^\delta(\cdot,x) \in C([0,T],\mathbb{R})$.

Then, for all $x_0 \in \mathbb{R}^N$, all $0 < t < T$ and all $R > 0$,

$$\int_{B(x_0,R)} |u-v|(t) \leq \int_{B(x_0,R+Lt)} K(t) * S_{|\Pi|}(t) * |u_0 - v_0|\,,$$

where L is a Lipschitz constant of f on $[-m_0(T), m_0(T)]$.

Proof. The proof mainly follows [1, Section 4].

Define $\psi(t,s,x,y) := \theta_\nu(s-t)\rho_\mu(y-x)\phi(t,x)$, where $\theta_\nu \in C_c^\infty((0,\nu))$ and $\rho_\mu \in C_c^\infty(B(0,\mu))$ are two approximate units and $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^N)$ is non-negative. By using the so-called *doubling variables technique*, we see that [1, inequality (4.3)] holds true with an additional term, namely

$$-\int_0^\infty \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(t,s,x,y)\mathrm{sgn}(u(t,x)-v(s,y)) \times \\ ((\Pi * u)(t,x) - (\Pi * v)(s,y))\, dydxdsdt\,.$$

By bounding this term from above, we see that [1, inequality (4.6)] holds true with the additional term

$$A_{\nu,\mu} := \int_0^\infty \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \theta_\nu(s-t)\rho_\mu(y-x)\phi(t,x) \times \\ |(\Pi * u)(t,x) - (\Pi * v)(s,y)|\, dydxdsdt\,.$$

Since $\Pi * v$ is locally integrable, it follows from classical properties of approximate units that, as $(\nu,\mu) \to (0,0)$,

$$A_{\nu,\mu} \to \int_0^\infty \int_{\mathbb{R}^N} \phi(t,x)|\Pi * (u-v)|(t,x)\, dxdt\,,$$

which is bounded from above by

$$\int_0^\infty \int_{\mathbb{R}^N} \phi\,(|\Pi| * |u-v|) = \int_0^\infty \int_{\mathbb{R}^N} |u-v|\,(|\tilde{\Pi}| * \phi)\,,$$

where $\tilde{\Pi}(x) := \Pi(-x)$. Then, we collect the analogues of [1, (4.11)] with this additional term: for all non-negative $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^N)$ such that $\mathrm{Supp}\,\phi \subset [0,T]\times\overline{B(0,R)}$, we have

$$\int_0^\infty \int_{\mathbb{R}^N} |u-v| \left(\partial_t\phi + L|\nabla\phi| + |\tilde{\Pi}| * \phi - g_\lambda[\phi]\right) \\ + \int_{\mathbb{R}^N} |u_0 - v_0|\phi(0,\cdot) \geq 0\,, \quad (5.1)$$

with L a Lipschitz constant of f on $[-m(T), m(T)]$, where

$$m(T) := \max\{\|u\|_{L^\infty((0,T)\times\mathbb{R}^N)}, \|v\|_{L^\infty((0,T)\times\mathbb{R}^N)}\}\,. \quad (5.2)$$

Let us define $\Lambda(t) := K(t) * S_{|\tilde{\Pi}|}(t)$, so that the solution to $\partial_t v - |\tilde{\Pi}| * v + g_\lambda[v] = 0$ with initial condition v_0 is given by $\Lambda(t) * v_0$. Now, we fix $x_0 \in \mathbb{R}^N$ and $M > LT$. Let $\gamma \in C_c^\infty([0,\infty))$ be non-negative, non-increasing and equal to 1 on $[0, M]$, and let $\Theta \in C_c^\infty([0,T))$. We define

$$\phi(t,x) := \begin{cases} \Theta(t)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)]\,(x) & \text{if } 0 \le t < T\,, \\ 0 & \text{if } t \ge T\,. \end{cases} \tag{5.3}$$

Note that $(t,x) \in [0,T] \times \mathbb{R}^N \mapsto \gamma(|x - x_0| + Lt)$ belongs to $C_c^\infty([0,T] \times \mathbb{R}^N)$ (it is equal to 1 on a neighbourhood of $[0,T] \times \{x_0\}$, so the non-smoothness of $|\cdot|$ at 0 does not play any role). Therefore, the definition of Λ implies that the function ϕ belongs to $C_b^\infty([0,\infty) \times \mathbb{R}^N)$, is non-negative and belongs to $L^1(0,T;W^{2,1}(\mathbb{R}^N))$. Hence, as in [1], we claim that, even if its support is not compact, ϕ can be used as a test function in (5.1).

We have $\partial_t(\Lambda(T-t)) + |\tilde{\Pi}| * \Lambda(T-t) - g_\lambda[\Lambda(T-t)] = 0$ and $g_\lambda[a * b] = g_\lambda[a] * b$. Therefore we see that, for all $(t,x) \in (0,T) \times \mathbb{R}^N$,

$$\begin{aligned}\left(\partial_t \phi + |\tilde{\Pi}| * \phi - g_\lambda[\phi]\right)(t,x) = \Theta'(t)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)]\,(x) \\ + L\Theta(t)\ \left[\Lambda(T-t) * \gamma'(|\cdot - x_0| + Lt)\right](x)\,.\end{aligned} \tag{5.4}$$

Since $\Lambda \ge 0$ and $\gamma' \le 0$ we also have

$$\begin{aligned} |\nabla \phi(t,x)| &= \left| \Theta(t) \left[\Lambda(T-t) * \frac{\cdot - x_0}{|\cdot - x_0|} \gamma'(|\cdot - x_0| + Lt) \right](x) \right| \\ &\le -\Theta(t)\ \left[\Lambda(T-t) * \gamma'(|\cdot - x_0| + Lt)\right](x)\,. \end{aligned} \tag{5.5}$$

Summing (5.4) and (5.5) we obtain

$$(\partial_t \phi + L|\nabla\phi| + |\tilde{\Pi}| * \phi - g_\lambda[\phi])(t,x) \le \Theta'(t)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)]\,(x)\,,$$

and, injecting this result into (5.1), we see that

$$\begin{aligned} &\int_0^T -\Theta'(t) \left(\int_{\mathbb{R}^N} |u - v|(t,\cdot)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)] \right) dt \\ &\le \int_{\mathbb{R}^N} \Theta(0) |u_0 - v_0|\ [\Lambda(T) * \gamma(|\cdot - x_0|)]\ . \end{aligned} \tag{5.6}$$

The above estimate is enough to prove the uniqueness of the entropy solution to ((1.4),(1.2)). Indeed, assume that $u_0 \equiv v_0$. We select a non-increasing $\Theta \in C_c^\infty([0,T))$ such that $\Theta'(t) = -1$ for all $0 \le t \le T/2$; then (5.6) yields

$$\int_{\mathbb{R}^N} |u - v|(t,\cdot)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)] = 0\,, \tag{5.7}$$

for all $0 \le t \le T/2$. We notice that, for all $s > 0$, $\Lambda(s) = K(s) + K(s) * (S_{|\tilde{\Pi}|}(s) - \delta_0) \ge K(s) > 0$ on $\mathbb{R}^N$. Moreover, for all $t \in [0,T]$, $\gamma(|\cdot - x_0| + Lt)$

is non-negative on $\mathbb{R}^N$ and positive on a ball around x_0; we deduce that, for all $t \in (0,T)$, $\Lambda(T-t) * [\gamma(|\cdot - x_0| + Lt)] > 0$ on $\mathbb{R}^N$. Hence, equation (5.7) shows that $u = v$ on $[0,T/2] \times \mathbb{R}^N$; this relation being valid for any T, this concludes the proof that the entropy solution is unique. As a by-product, we notice that this entropy solution is the one constructed in Section 4, and therefore that it belongs to $C([0,\infty); L^1_{\rm loc}(\mathbb{R}^N))$ and satisfies $\|u\|_{L^\infty((0,T)\times\mathbb{R}^N))} \leq e^{\|\Pi\|_1 T}\|u_0\|_{L^\infty(\mathbb{R}^N)}$; hence, $m(T)$ defined in (5.2) is bounded from above by $m_0(T)$ defined in Proposition 5.1.

We now conclude the proof of Proposition 5.1. For $0 < \nu < T$, let $\theta_\nu \in C_c^\infty((0,\nu))$ be an approximate unit. Hence, Θ given by

$$\Theta(t) := \int_t^\infty \theta_\nu(T-s)\, ds$$

belongs to $C_c^\infty([0,T))$ and satisfies $\Theta(0) = 1$. From (5.6), we infer

$$\int_0^T \theta_\nu(T-t) \left(\int_{\mathbb{R}^N} |u-v|(t,\cdot)\ [\Lambda(T-t) * \gamma(|\cdot - x_0| + Lt)] \right) dt$$
$$\leq \int_{\mathbb{R}^N} |u_0 - v_0|\ [\Lambda(T) * \gamma(|\cdot - x_0|)]\ . \tag{5.8}$$

The function $t \in [0,T] \mapsto \Lambda(T-t) * \gamma(|\cdot - x_0| + Lt) \in L^1(\mathbb{R}^N)$ is continuous ([7]); moreover, by the continuity of the entropy solutions u, v with values in $L^1_{\rm loc}(\mathbb{R}^N)$ (proved above) and their L^∞ bound, we see that $t \in [0,\infty) \mapsto |u-v|(t,\cdot)$ is continuous with values in $L^\infty(\mathbb{R}^N)$ weak-$*$. We can therefore pass to the limit $\nu \to 0$ in (5.8) to find

$$\int_{\mathbb{R}^N} |u-v|(T,\cdot)\gamma(|\cdot - x_0| + LT)$$
$$\leq \int_{\mathbb{R}^N} |u_0 - v_0|\ \left[K(T) * S_{|\tilde{\Pi}|}(T) * \gamma(|\cdot - x_0|)\right]$$
$$= \int_{\mathbb{R}^N} \gamma(|\cdot - x_0|)\ \left[K(T) * S_{|\Pi|}(T) * |u_0 - v_0|\right]\ , \tag{5.9}$$

where we have used the fact that $K(T)$ is even. To conclude we approximate in $L^1(\mathbb{R}^N)$ the characteristic function of the ball $B(x_0, R+LT)$ by functions of the form $\gamma(|\cdot - x_0|)$, with γ as above. Passing to such approximation limit in (5.9) we collect

$$\int_{B(x_0,R)} |u-v|(T) \leq \int_{B(x_0,R+LT)} K(T) * S_{|\Pi|}(T) * |u_0 - v_0|\,,$$

which concludes the proof of Proposition 5.1. ■

[7] $\Lambda : (0,\infty) \to L^1(\mathbb{R}^N)$ is continuous and is an approximate unit as $t \to 0$, and the function $(t,x) \in [0,\infty) \times \mathbb{R}^N \mapsto \gamma(|\cdot - x_0| + Lt)$ is continuous with compact support.

6 Regularising effect for $1 < \lambda \le 2$

In this section we assume $1 < \lambda \le 2$ and we prove Theorem 2.6.

6.1 Duhamel's formula for the entropy solution

Denoting by u^δ the function constructed by the splitting method in Section 4, we first obtain an integral equation on u^δ which, by letting $\delta \to 0$, shows that the entropy solution $u = \lim_{\delta\to 0} u^\delta$ satisfies the Duhamel's formula corresponding to $\partial_t u + \mathcal{G}[u] = -\operatorname{div}(f(u))$. More precisely the following holds.

Proposition 6.1. *Let u be the entropy solution to* (1.4) *with initial data $u_0 \in L^\infty(\mathbb{R}^N)$. Then, for all $t > 0$,*

$$\begin{aligned} u(t) &= (K(t) * S_{-\Pi}(t)) * u_0 \\ &\quad - \int_0^t \nabla (K(t-s) * S_{-\Pi}(t-s)) * f(u(s))\, ds\,, \end{aligned} \tag{6.1}$$

where $h^{(1)} * h^{(2)} := \sum_{i=1}^N h_i^{(1)} * h_i^{(2)}$ if $h^{(j)} = (h_1^{(j)}, ..., h_N^{(j)}) : \mathbb{R}^N \to \mathbb{R}^N$, $j = 1, 2$.

Proof. Let us first recall that $K^{[2]}(t) := K(2t)$ and ${S_{-\Pi}}^{[2]}(t) := S_{-\Pi}(2t)$. Assume that $2n\delta < t \le (2n+1)\delta$, for some $n \ge 0$. Since u^δ is the entropy solution to (4.1) on $(2n\delta, t]$ and since $\lambda > 1$, we can write the following Duhamel's formula (see [14])

$$u^\delta(t) = K^{[2]}(t - 2n\delta) * u^\delta(2n\delta) - 2\int_{2n\delta}^t \nabla K^{[2]}(t-s) * f(u^\delta(s))\, ds\,. \tag{6.2}$$

Now, if $n \neq 0$ we go further in the past. On $((2n-1)\delta, 2n\delta]$, u^δ solves (4.2) so that

$$u^\delta(2n\delta) = {S_{-\Pi}}^{[2]}(\delta) * u^\delta((2n-1)\delta)\,, \tag{6.3}$$

which, combined with (6.2), yields

$$\begin{aligned} u^\delta(t) &= K^{[2]}(t-2n\delta) * {S_{-\Pi}}^{[2]}(\delta) * u^\delta((2n-1)\delta) \\ &\quad -2\int_{2n\delta}^t \nabla K^{[2]}(t-s) * f(u^\delta(s))\, ds\,. \end{aligned} \tag{6.4}$$

Another Duhamel's formula for u^δ on $(2(n-1)\delta, (2n-1)\delta]$ yields

$$\begin{aligned} u^\delta((2n-1)\delta) &= K^{[2]}(\delta) * u^\delta(2(n-1)\delta) \\ &\quad -2\int_{2(n-1)\delta}^{(2n-1)\delta} \nabla K^{[2]}((2n-1)\delta - s) * f(u^\delta(s))\, ds\,. \end{aligned}$$

By plugging this into (6.4) and using the semi-group properties of K and $S_{-\Pi}$ (see Proposition 3.1), we deduce

$$\begin{aligned} u^\delta(t) &= K^{[2]}(t-2n\delta+\delta) * S_{-\Pi}{}^{[2]}(\delta) * u^\delta(2(n-1)\delta) \\ &\quad -2\int_{2n\delta}^{t} \nabla K^{[2]}(t-s) * f(u^\delta(s))\, ds \\ &\quad -2\int_{2(n-1)\delta}^{2(n-1)\delta+\delta} \nabla K^{[2]}(t-s-\delta) * S_{-\Pi}{}^{[2]}(\delta) * f(u^\delta(s))\, ds \end{aligned} \quad (6.5)$$

Iterating $n-1$ more times the process from (6.3) to (6.5), we arrive at

$$\begin{aligned} u^\delta(t) = K^{[2]}(t-n\delta) * S_{-\Pi}{}^{[2]}(n\delta) * u_0 - 2\int_{2n\delta}^{t} \nabla K^{[2]}(t-s) * f(u^\delta(s))\, ds \\ -\sum_{k=1}^{n} 2\int_{2(n-k)\delta}^{2(n-k)\delta+\delta} \nabla K^{[2]}(t-s-k\delta) * S_{-\Pi}{}^{[2]}(k\delta) * f(u^\delta(s))\, ds\,. \end{aligned} \quad (6.6)$$

Let a_δ^i, $i=1,...,4$, be the functions defined, for all $n\geq 0$ and all $0\leq k\leq n$, by

$$a_\delta^1(t) := \begin{cases} 2(t-n\delta) & \text{if } 2n\delta \leq t < (2n+1)\delta \\ 2((2n+1)\delta - n\delta) & \text{if } (2n+1)\delta \leq t < 2(n+1)\delta\,, \end{cases}$$

$$a_\delta^2(t) := \begin{cases} 2(n\delta) & \text{if } 2n\delta \leq t < (2n+1)\delta \\ 2(n\delta + t - (2n+1)\delta) & \text{if } (2n+1)\delta \leq t < 2(n+1)\delta\,, \end{cases}$$

$$a_\delta^3(t,s) := \begin{cases} 2(t-s-k\delta) & \text{if } \begin{cases} 2n\delta \leq t < (2n+1)\delta \text{ and} \\ 2(n-k)\delta \leq s < 2(n-k)\delta+\delta \end{cases} \\ 2((2n+1)\delta - s - k\delta) & \text{if } \begin{cases} (2n+1)\delta \leq t < 2(n+1)\delta \text{ and} \\ 2(n-k)\delta \leq s < 2(n-k)\delta+\delta\,, \end{cases} \\ t-s & \text{if } \begin{cases} 2n\delta \leq t < 2(n+1)\delta \text{ and} \\ 2(n-k)\delta+\delta \leq s < 2(n-k)\delta+2\delta\,, \end{cases} \end{cases}$$

$$a_\delta^4(t,s) := \begin{cases} 2(k\delta) & \text{if } \begin{cases} 2n\delta \leq t < (2n+1)\delta \text{ and} \\ 2(n-k)\delta \leq s < 2(n-k)\delta+2\delta \end{cases} \\ 2(k\delta + t - (2n+1)\delta) & \text{if } \begin{cases} (2n+1)\delta \leq t < 2(n+1)\delta \text{ and} \\ 2(n-k)\delta \leq s < 2(n-k)\delta+2\delta\,. \end{cases} \end{cases}$$

Case-by-case study show that the following pointwise estimates hold:

$$\begin{aligned} &|a_\delta^1(t)-t| \leq \delta\,, \quad |a_\delta^2(t)-t| \leq \delta\,, \quad |a_\delta^3(t,s)-(t-s)| \leq 2\delta \\ &\text{and} \quad |a_\delta^4(t,s)-(t-s)| \leq 2\delta\,. \end{aligned}$$

Moreover (6.6) is recast as

$$\begin{aligned} u^\delta(t) &= K(a_\delta^1(t)) * S_{-\Pi}(a_\delta^2(t)) * u_0 \\ &\quad - \int_0^t 2I_\delta(s)\nabla K(a_\delta^3(t,s)) * S_{-\Pi}(a_\delta^4(t,s)) * f(u^\delta(s))\, ds\,, \end{aligned} \tag{6.7}$$

with I_δ the characteristic function of $\cup_{n\geq 0}[2n\delta,(2n+1)\delta)$ ([8]).

If $(2n+1)\delta < t \leq 2(n+1)\delta$ for some $n \geq 0$ then, writing $u^\delta(t) = {S_{-\Pi}}^{[2]}(t-(2n+1)\delta) * u^\delta((2n+1)\delta)$ and using (6.7) for $t=(2n+1)\delta$, we see — by our choice of the functions a_δ^i — that (6.7) remains valid.

We aim at letting $\delta \to 0$ in (6.7). From our pointwise estimates on the functions a_δ^i and item (viii) in Proposition 3.1, we see that, for all $t>0$,

$$K(a_\delta^1(t)) * S_{-\Pi}(a_\delta^2(t)) \to K(t) * S_{-\Pi}(t) \quad \text{in } L^1(\mathbb{R}^N)\,,$$

and that, for all $0<s<t$,

$$\nabla K(a_\delta^3(t,s)) * S_{-\Pi}(a_\delta^4(t,s)) \to \nabla K(t-s) * S_{-\Pi}(t-s) \quad \text{in } L^1(\mathbb{R}^N)^N\,.$$

Recalling that $u^\delta \to u$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and that u^δ remains bounded in $L^\infty((0,T)\times\mathbb{R}^N)$ we also get that, for all $s>0$, $f(u^\delta(s)) \to f(u(s))$ in $L^\infty(\mathbb{R}^N)$ weak-$*$. Combining this with the above limit yields that, for all $0<s<t$,

$$\begin{aligned} Z_\delta(t,s) := \nabla K(a_\delta^3(t,s)) * S_{-\Pi}(a_\delta^4(t,s)) * f(u^\delta(s)) \\ \to \nabla K(t-s) * S_{-\Pi}(t-s) * f(u(s))\,. \end{aligned} \tag{6.8}$$

Moreover, by Young's inequality for the convolution and the integrability property of ∇K (see item (ii) in Proposition 3.1), we see that

$$\|Z_\delta(t,s)\|_{C_b(\mathbb{R}^N)} \leq C\, a_\delta^3(t,s)^{-1/\lambda}\,,$$

where, here and in the following, C does not depend on δ, t or s and may change from place to place. Studying separately the case $k=1$ in the first line defining a_δ^3, the case $k=0$ in the second line defining a_δ^3 and the other cases ($k\neq 1$ in the first line, $k\neq 0$ in the second, $k\geq 0$ in the third), one can find a lower bound on a_δ^3 which shows that

$$\begin{aligned} a_\delta^3(t,s)^{-1/\lambda} \leq &\; \frac{C\mathbf{1}_{[2(n-1)\delta,2(n-1)\delta+\delta)}(s)}{(t-s-\delta)^{1/\lambda}} \\ &+ \frac{C\mathbf{1}_{[2n\delta,2n\delta+\delta)}(s)}{((2n+1)\delta-s)^{1/\lambda}} + \frac{C}{(t-s)^{1/\lambda}}\,, \end{aligned} \tag{6.9}$$

where n is taken such that $2n\delta \leq t < 2(n+1)\delta$. The integral for $s\in(0,t)$ of the two first functions in the right-hand side member of (6.9) is bounded by

[8] Note that the definition of $a_\delta^3(t,s)$ for $2(n-k)\delta+\delta \leq s < 2(n-k)\delta+2\delta$ does not play any role in (6.7), and the choice $a_\delta^3(t,s) = t-s$ in these cases is made by convenience.

$C\delta^{1-\frac{1}{\lambda}}$ and thus tends to 0 as $\delta \to 0$. The estimate (6.9) therefore shows that the sequence $(a_\delta^3(t,\cdot)^{-1/\lambda})_{\delta\to 0}$ is equi-integrable on $(0,t)$ and, using Vitali's Theorem, we conclude that the convergence in (6.8) also holds in $L^1(0,t)$, pointwise on $\mathbb{R}^N$.

Since $2I_\delta \to 1$ in $L^\infty(0,\infty)$ weak-$*$, the above considerations allow us to pass to the limit $\delta \to 0$ in (6.7). Hence, the entropy solution u to (1.4) satisfies the Duhamel's formula (6.1). ■

6.2 Regularity of the entropy solution: proof of Theorem 2.6

Let us recall that, in the case where $\Pi \equiv 0$, a regularising effect is proved for $1 < \lambda \le 2$ in [14]. The authors take advantage of the Duhamel's formula involving K rather than $K * S_{-\Pi}$. Since the regularity and integrability properties of $K * S_{-\Pi}$ and $\nabla(K * S_{-\Pi})$ are similar to the properties of K and ∇K (see Proposition 3.1), we can reproduce the techniques used in the proof of [14, Proposition 5.1, Theorem 5.2]. Therefore the entropy solution u to (1.4) is indefinitely derivable with respect to x on $(0,\infty)\times\mathbb{R}^N$. Moreover, for all $0 < a < T$ and all $(i_1,...,i_N) \in \mathbb{N}^N$, we have $\partial_{x_1}^{i_1}...\partial_{x_N}^{i_N} u \in C_b((a,T)\times\mathbb{R}^N)$. Finally, the entropy formulation (2.3) with $\eta(s) = \pm s$ shows that u satisfies (1.4) in the distributional sense; hence the spatial regularity of u ensures, by a bootstrap argument, that it is also regular in time.

Theorem 2.6 is proved.

7 Generalizations

Here we handle two generalisations of (1.4) by the preceding methods.

7.1 Dirac masses in Π

Our results remain true if Assumption 1 is replaced by Assumption 2, i.e. if there exists $c \in \mathbb{R}$ such that $\Pi := \mathcal{F}^{-1}(|\cdot|^\lambda(H(\cdot)-1)) \in c\delta_0 + L^1(\mathbb{R}^N)$. This allows to consider the cases where $|\xi|^\lambda(H(\xi)-1) \to c$ quickly enough as $|\xi| \to \infty$: for example, it is satisfied if $|\cdot|^\lambda(H(\cdot)-1) - c \in W^{N+1,1}(\mathbb{R}^N)$ (see also the appendix for a less demanding property on H, which implies Assumption 2).

Defining $\Pi_1 := \Pi - c\delta_0 \in L^1(\mathbb{R}^N)$, equation (1.4) then becomes

$$\partial_t u + \mathrm{div}(f(u)) + g_\lambda[u] + \Pi_1 * u + cu = 0\,.$$

Thus Assumption 2 consists in adding a linear reaction term cu into the considered equation.

In terms of mathematical study, the replacement of Assumption 1 by Assumption 2 brings minor changes (some of which are listed below) and all the preceding theorems remain valid.

(i) the term $\Pi * u$ is changed into $\Pi_1 * u + cu$,

(ii) the estimate (4.3) becomes $||u^\delta(t)||_\infty \leq e^{-ct} e^{||\Pi_1||_1 t} ||u_0||_\infty$ (and thus the multiplicative term e^{-ct} must be applied to all the estimates derived from (4.3)),

(iii) on $((2n-1)\delta, 2n\delta]$ we have $\partial_t u^\delta + 2\,\Pi_1 * u^\delta + 2cu^\delta = 0$ so that, if $v^\delta := e^{2ct} u^\delta$, equality $\partial_t(v^\delta - \mathcal{T}_h v^\delta) + 2\,(\Pi_1 - \mathcal{T}_h \Pi_1) * v^\delta = 0$ holds. Hence, if $w_T(h) := 2\|\Pi_1 - \mathcal{T}_h\Pi_1\|_1 e^{|c|T} e^{\|\Pi_1\|_1 T} \|u_0\|_\infty$, we see that (4.5) holds true for v^δ in place of u^δ. Coming back to u^δ the estimate (4.5) is changed into

$$\begin{aligned} &|u^\delta - \mathcal{T}_h u^\delta|(2n\delta, x) \\ &\qquad \leq \; e^{-2c2n\delta} \omega_T(h)\delta + e^{-2c\delta} |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta, x) \\ &\qquad \leq \; e^{2|c|T} \omega_T(h)\delta + e^{2|c|\delta} |u^\delta - \mathcal{T}_h u^\delta|((2n-1)\delta, x)\,. \end{aligned}$$

Therefore (4.6) is valid with $\omega_T(h)$ multiplied by $e^{2|c|T}$ and $K^{[2]}(t - 2n\delta)$ by $e^{2|c|\delta}$; after having cumulated all the time steps, the final inequality (4.8) is valid with $\omega_T(h)$ and $K^{[2]}(s)$ multiplied by $e^{2|c|T}$ and the end of the translation estimates follows,

(iv) the semi-groups $S_{-\Pi}(t)$, $S_{|\tilde{\Pi}|}(t)$ and $S_{|\Pi|}(t)$ are replaced by $e^{ct} S_{-\Pi_1}(t)$, $e^{|c|t} S_{|\tilde{\Pi}_1|}(t)$ and $e^{|c|t} S_{|\Pi_1|}(t)$.

7.2 Time-dependent Π

It is also possible to handle the case where Π depends on t, for example $\Pi \in C([0,\infty); L^1(\mathbb{R}^N))$. In this case, the solution to $\partial_t u(t) + \Pi(t) * u(t) = 0$ with initial data $u(t_0) = u_0$ is no longer given by a semi-group but by the flow $S_{-\Pi}(t; t_0) * u_0$ with

$$S_{-\Pi}(t; t_0) := \delta_0 + \sum_{n \geq 1} \frac{1}{n!} \left(\int_{t_0}^t -\Pi(s)\, ds \right)^{*(n)}.$$

Here again the adaptation of the techniques and estimates are quite straightforward; for example, the estimate (4.3) becomes

$$||u^\delta(t)||_\infty \leq e^{2 \int_{[0,t] \cap J_\delta} ||\Pi(s)||_1\, ds} ||u_0||_\infty\,.$$

The existence and uniqueness of the entropy solution (Theorem 2.3) are valid under the assumption $\Pi \in C([0,\infty); L^1(\mathbb{R}^N))$, and the regularising effect (Theorem 2.6) under the assumption $\Pi \in C^\infty([0,\infty); L^1(\mathbb{R}^N))$.

A Appendix: the mathematical assumptions in the physical context

We come back here to the physical model presented in Section 1. As seen in [10] and [12], the function W has the integral representation $W(is) = \int_0^\infty w_1(\xi)e^{-is\xi}d\xi + \int_0^\infty (1+is\xi)w_2(\xi)e^{-is\xi}d\xi$, with w_1 and w_2 regular functions such that $w_1(0)+w_2(0) = ib$. The numerical approximations [10] of w_1 and w_2 exhibit rapid convergence to 0 at infinity. Hence, integrating-by-part, one can find asymptotic expansions of W and its derivatives which show that

$$\begin{array}{l} \lim_{s\to\infty} s(sW(is)-b) \text{ exists, is finite and, for } k=1,2, \\ \left|\frac{d^k}{ds^k}(sW(is))\right| + \left|\frac{d^k}{ds^k}(s(sW(is)-b))\right| = \mathcal{O}\left(\frac{1}{s}\right) \text{ as } s\to\infty. \end{array} \tag{A.1}$$

We prove here that, thanks to this property of W, the function $H(\xi) = \sqrt{1+W(i|\xi|)}$ is such that

$$\mathcal{F}^{-1}(|\cdot|(H(\cdot)-1)) \in \frac{b}{2}\delta_0 + L^1(\mathbb{R}). \tag{A.2}$$

In other words, H satisfies Assumption 2 with $\lambda = 1$ ([9]), and thus our preceding study in Sections 4 and 5 covers the physical model under consideration.

We take a cut-off function $\chi \in C_c^\infty(\mathbb{R})$, equal to 1 on $[-1,1]$, and we write

$$\begin{aligned} |\xi|(H(\xi)-1) &= |\xi|\frac{W(i|\xi|)}{\sqrt{1+W(i|\xi|)}+1} \\ &= |\xi|\chi(\xi)\frac{W(i|\xi|)}{\sqrt{1+W(i|\xi|)}+1} \\ &\quad +|\xi|(1-\chi(\xi))\frac{W(i|\xi|)}{\sqrt{1+W(i|\xi|)}+1} \\ &=: T_1(\xi)+T_2(\xi). \end{aligned} \tag{A.3}$$

We are first concerned with T_1. By regularity of W, an asymptotic expansion of $\frac{W(is)}{\sqrt{1+W(is)}+1}$ around $s=0$ shows that

$$T_1(\xi) = d|\xi|\chi(\xi) + \xi^2\chi(\xi)\gamma(|\xi|),$$

[9]In [10], [11], W is actually a complex-valued function and we should take the real part of $\sqrt{1+W}$ when defining H. However, in order to simplify the presentation, we will omit this and study the "full" $H = \sqrt{1+W}$ (the real part of this expression cannot have a worst behaviour than the expression itself). Note also that, in the physical context, W seems to be small enough to ensure that a smooth determination of the complex square root can be chosen, so that H can be considered smooth outside $\xi = 0$.

with d a constant and γ regular. By Lemma 3.2, we see that

$$\mathcal{F}^{-1}(|\cdot|\chi(\cdot)) = \mathcal{F}^{-1}(|\cdot|\mathcal{F}(\mathcal{F}^{-1}(\chi))(\cdot)) = g_1[\mathcal{F}^{-1}(\chi)] \in L^1(\mathbb{R}),$$

since $\mathcal{F}^{-1}(\chi) \in \mathcal{S}(\mathbb{R})$. Moreover, the function $\xi \mapsto \xi^2\chi(\xi)\gamma(|\xi|)$ belongs to $W^{2,1}(\mathbb{R})$ (the singularities at 0 appearing, because of $|\xi|$, in the first and second derivatives of $\gamma(|\xi|)$ are compensated by the term ξ^2) and its inverse Fourier transform is therefore integrable. Hence,

$$\mathcal{F}^{-1}(T_1) \in L^1(\mathbb{R}). \tag{A.4}$$

We now handle T_2. Since $W(is) \sim b/s$ as $s \to \infty$, we see that $T_2(\xi) \to b/2$ as $|\xi| \to \infty$. Moreover, for $|\xi|$ large enough (such that $\chi(\xi) = 0$), we have

$$T_2(\xi) - \frac{b}{2} = \frac{2(|\xi|W(i|\xi|) - b) - b(\sqrt{1 + W(i|\xi|)} - 1)}{2(\sqrt{1 + W(i|\xi|)} + 1)}.$$

From this relation we understand that $T_2(\xi) - \frac{b}{2}$ behaves "at worst" at ∞ as $|\xi|W(i|\xi|) - b$ or $W(i|\xi|)$. More precisely, since $T_2 - \frac{b}{2}$ is regular at $\xi = 0$, we can write

$$T_2(\xi) - \frac{b}{2} = \frac{\mu(\xi)}{|\xi|} + \alpha(\xi),$$

with $\alpha \in C_c^\infty(\mathbb{R})$ and μ regular, vanishing on a neighbourhood of 0, having limits at $\pm\infty$ and satisfying $|\mu'(\xi)| + |\mu''(\xi)| = \mathcal{O}(1/|\xi|)$ at infinity ([10]). Lemma A.1 below thus ensures that $\mathcal{F}^{-1}\left(T_2 - \frac{b}{2}\right) \in L^1(\mathbb{R})$, i.e. that

$$\mathcal{F}^{-1}(T_2) \in \frac{b}{2}\delta_0 + L^1(\mathbb{R}). \tag{A.5}$$

Gathering (A.4), (A.5) and (A.3), we infer that (A.2) holds true, thus concluding the proof that, in the considered framework, Assumption 2 holds.

Lemma A.1. *Let $\mu \in C_b^1(\mathbb{R})$ be such that $\mu = 0$ on a neighbourhood of 0 and $\mu'(\xi) = \mathcal{O}(1/|\xi|)$ as $|\xi| \to \infty$. Then $\mathcal{F}^{-1}(\frac{\mu(\cdot)}{|\cdot|}) \in L^1_{\text{loc}}(\mathbb{R})$.*

Moreover, if $\mu \in C_b^2(\mathbb{R})$ and if $\frac{\mu''(\cdot)}{|\cdot|} \in L^1(\mathbb{R})$, then $\mathcal{F}^{-1}(\frac{\mu(\cdot)}{|\cdot|}) \in L^1(\mathbb{R})$.

Proof. Let $A > 0$ and $f_A := \mathcal{F}^{-1}(\frac{\mu(\cdot)}{|\cdot|}\mathbf{1}_{[-A,A]}(\cdot))$. Then $f_A \in L^\infty(\mathbb{R})$ and, since $\frac{\mu(\cdot)}{|\cdot|}\mathbf{1}_{[-A,A]}(\cdot) \to \frac{\mu(\cdot)}{|\cdot|}$ in $\mathcal{S}'(\mathbb{R})$ as $A \to \infty$, we have $f_A \to f := \mathcal{F}^{-1}(\frac{\mu(\cdot)}{|\cdot|})$ in $\mathcal{S}'(\mathbb{R})$ and thus also in $\mathcal{D}'(\mathbb{R})$. We prove below that f_A converges a.e. as $A \to \infty$ and that $(f_A)_{A>0}$ stays bounded by a function $g \in L^1_{\text{loc}}(\mathbb{R})$: the dominated convergence theorem then ensures that f_A converges in $L^1_{\text{loc}}(\mathbb{R})$ and thus that $f \in L^1_{\text{loc}}(\mathbb{R})$.

[10] This is where (A.1) is used: $\mu(\xi)$ and its derivatives behave at infinity "at worst" like $|\xi|(|\xi|W(i|\xi|) - b)$ or $|\xi|W(i|\xi|)$ and their derivatives.

To prove the convergence and boundedness of f_A, we take $a > 0$ such that $\mu = 0$ on $[-a, a]$ and we write, for $x \neq 0$,

$$
\begin{aligned}
f_A(x) &= \int_{|\xi|\le A} \frac{\mu(\xi)}{|\xi|} e^{2i\pi x\xi}\, d\xi \\
&= \int_{a\le|\xi|\le\min(A,1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2i\pi x\xi}\, d\xi \\
&\quad +\mathbf{1}_{\{|x|A\ge1\}} \int_{1/|x|\le|\xi|\le A} \frac{\mu(\xi)}{|\xi|} e^{2i\pi x\xi}\, d\xi\,.
\end{aligned}
$$

Using, in the second integral sign, the change of variable $z = x\xi$ and an integration by parts, we find

$$
\begin{aligned}
f_A(x) &= \int_{a\le|\xi|\le\min(A,1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2i\pi x\xi}\, d\xi \\
&\quad +\mathbf{1}_{\{|x|A\ge1\}} \int_{1\le|z|\le|x|A} \frac{\mu(z/x)}{|z|} e^{2i\pi z}\, dz \\
&= \int_{a\le|\xi|\le\min(A,1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2i\pi x\xi}\, d\xi \\
&\quad +\mathbf{1}_{\{|x|A\ge1\}} \left[\frac{\mu(\frac{|x|A}{x})}{|x|A} \frac{e^{2i\pi|x|A}}{2i\pi} - \frac{\mu(\frac{-|x|A}{x})}{|x|A} \frac{e^{-2i\pi|x|A}}{2i\pi} \right] \\
&\quad -\mathbf{1}_{\{|x|A\ge1\}} \frac{\mu(\frac{1}{x}) - \mu(\frac{-1}{x})}{2i\pi} \\
&\quad -\mathbf{1}_{\{|x|A\ge1\}} \int_{1\le|z|\le|x|A} \frac{e^{2i\pi z}}{2i\pi} \left(\frac{\frac{1}{x}\mu'(\frac{z}{x})}{|z|} - \frac{\mu(\frac{z}{x})\mathrm{sgn}(z)}{z^2} \right) dz\,.
\end{aligned}
$$

Since μ is bounded and $\mu'(\xi) = \mathcal{O}(1/|\xi|)$ as $|\xi| \to \infty$, the integrand in the last integral sign is bounded by C/z^2, with C not depending on x or A. Therefore the above expression of $f_A(x)$ shows that it converges, for all $x \neq 0$, as $A \to \infty$. Moreover, using again the above expression, we find $C > 0$, still not depending on x or A, such that

$$
\begin{aligned}
|f_A(x)| &\le \int_{a\le|\xi|\le1/|x|} \frac{C}{|\xi|}\, d\xi + C\mathbf{1}_{\{|x|A\ge1\}} + \mathbf{1}_{\{|x|A\ge1\}} \int_{1\le|z|} \frac{C}{z^2}\, dz \\
&\le 2C \ln\left(\frac{1}{a|x|}\right) + C + 2C =: g(x)\,.
\end{aligned}
$$

Since $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, the proof that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is complete.

We now assume that $\mu \in C^2_b(\mathbb{R})$ and that $\frac{\mu''(\cdot)}{|\cdot|} \in L^1(\mathbb{R})$. Then, noticing that

$$
\nu(\xi) := \frac{d^2}{d\xi^2} \frac{\mu(\xi)}{|\xi|} = \frac{\mu''(\xi)}{|\xi|} - 2\mathrm{sgn}(\xi)\frac{\mu'(\xi)}{\xi^2} + 2\mathrm{sgn}(\xi)\frac{\mu(\xi)}{\xi^3} = \frac{\mu''(\xi)}{|\xi|} + \mathcal{O}\left(\frac{1}{\xi^2}\right)
$$

as $\xi \to \infty$, we see that $\nu \in L^1(\mathbb{R})$ and thus that $\mathcal{F}^{-1}(\nu) \in L^\infty(\mathbb{R})$. Since $f(x) = \mathcal{F}^{-1}(\frac{\mu(\cdot)}{|\cdot|})(x) = \frac{1}{(2i\pi x)^2}\mathcal{F}^{-1}(\nu)(x)$, we infer that $f(x) = \mathcal{O}(1/x^2)$ at infinity so that $f \in L^1(\mathbb{R})$. ■

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