Rapid travelling waves in the nonlocal Fisher equation connect two unstable states

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Abstract

In this note, we give a positive answer to a question addressed in [8]. Precisely we prove that, for any kernel and any slope at the origin, there do exist travelling wave solutions (actually those which are "rapid") of the nonlocal Fisher equation that connect the two homogeneous steady states 0 (dynamically unstable) and 1. In particular this allows situations where 1 is unstable in the sense of Turing. Our proof does not involve any maximum principle argument and applies to kernels with *fat tails*.

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1. Introduction

In this work, we consider the nonlocal Fisher-KPP equation

$$\partial_t u = \partial_{xx} u + \mu u(1 - \phi * u) \qquad x \in \mathbb{R}\,, \quad t > 0\,, \tag{1}$$

where $\phi * u(x) := \displaystyle\int_{\mathbb{R}} u(x-y)\phi(y)\,dy$, ϕ is a given smooth kernel such that

$$\phi \geq 0\,, \quad \phi(0) > 0\,, \quad \int_{\mathbb{R}} \phi = 1\,, \quad \int_{\mathbb{R}} z^2 \phi(z)\,dz < \infty\,, \tag{2}$$

and $\mu > 0$ is identified as the "slope at the origin". We are interested in travelling waves solutions supported by the integro-differential equation (1). We are therefore looking after a speed $c \in \mathbb{R}$ and a smooth and bounded $u(x)$ such that

$$-cu' = u'' + \mu u(1 - \phi * u) \qquad x \in \mathbb{R}\,, \tag{3}$$

supplemented with the expected boundary conditions

$$u(-\infty) = 1\,, \qquad u(+\infty) = 0\,, \tag{4}$$

or, when necessary, the weaker boundary conditions

$$\liminf_{x \to -\infty} u(x) > 0\,, \qquad u(+\infty) = 0\,. \tag{5}$$

Local Fisher-KPP equation. If the kernel ϕ is replaced by the Dirac δ-function, then (1) reduces to

$$\partial_t u = \partial_{xx} u + \mu u(1-u)\,, \tag{6}$$

namely the classical Fisher-KPP equation [4], [7], for which $u \equiv 0$ is unstable and $u \equiv 1$ is stable. It is commonly used in the literature to model phenomena arising in population genetics, or in biological invasions. It is well known that the classical Fisher-KPP equation admits monotonic travelling wave solutions with the expected boundary conditions (4), for some semi-infinite interval $[c^* := 2\sqrt{\mu}, \infty)$ of admissible wave speeds. Moreover, these waves describe the long time behavior of solutions of (6) with compactly supported initial data or initial data with exponential decay.

Nonlocal Fisher-KPP equation. Let us turn back to the integro-differential equation (1). In population dynamics models, one can see the nonlinear term as the intra-specific competition for resources. Its nonlocal form indicates that individuals are competing with all other individuals, whatever their positions. For more details on nonlocal models, we refer to [6] and the references therein.

Again the uniform steady states of (1) are $u \equiv 0$ and $u \equiv 1$. Nevertheless, because of the nonlocal effect, the steady state 1 can be Turing unstable. In particular, this happens when the Fourier transform $\hat{\phi}$ changes sign and μ is large [5], [1]. The situation is therefore in contrast with (6). Hence, for travelling waves to be constructed, the authors in [2] have to ask not for the the expected behavior $u(-\infty) = 1$, but for the weaker condition $\liminf_{-\infty} u > 0$. More precisely, they prove the following.

Lemma 1 (Travelling waves constructed in [2]). *For all $c \geq c^* := 2\sqrt{\mu}$, there exists a travelling wave $(c, u) \in \mathbb{R} \times C_b^2(\mathbb{R})$ solution of (3), with $u > 0$ and with the weak boundary conditions (5).*

Moreover, there is $\mu_0 > 0$ such that, for all kernel ϕ, all $0 < \mu < \mu_0$, these waves actually satisfy $u(-\infty) = 1$.

Also, if the Fourier transform $\hat{\phi}$ is positive everywhere, then, for all $\mu > 0$, these waves actually satisfy $u(-\infty) = 1$.

When $\hat{\phi}$ takes negative values and when $\mu > 0$ is large, such results do not precise if the waves can approach the Turing unstable state 1 as $x \to -\infty$. By using numerical approximation, the authors in [8] observe such waves for the compactly supported kernel $\frac{1}{2}\mathbf{1}_{[-1,1]}$. It should be noted that for kernels with exponential decay one may use maximum principle arguments and then derive some monotonicity properties. Hence, it is proved in [3] that waves which are rapid enough are monotone and then approach 1 as $x \to -\infty$. Here, we allow kernels with *fat tails* which are quite relevant in applications. Precisely, we only assume that the second moment of ϕ is finite. The result of this note is to prove that, even for such kernels, rapid waves always connect 1 in $-\infty$. It reads as follows.

Theorem 2 (Rapid waves connect two unstable states). *Define*

$$\overline{c} = \overline{c}(\phi, \mu) := \mu \left(\int_{\mathbb{R}} z^2 \phi(z)\, dz \right)^{1/2} \left(\int_{\mathbb{R}} \phi(z)(1 - \mu \frac{z^2}{2})_+ \, dz \right)^{-1}.$$

Then the waves constructed in Lemma 1 with speed $c > \overline{c}$ actually satisfy $u(-\infty) = 1$.

Our proof does not use any maximum principle argument neither any monotonicity property of the wave. Therefore, Theorem 2 allows the possibility of non monotonic waves. It leans on L^2 estimates proved in Section 2. In Section 3, we complete the proof of Theorem 2 and conclude with remarks on the bistable case.

2. Investigating the behavior in $\pm\infty$

This section contains the main contribution of the present note. By rather elementary L^2 estimates, we find a sufficient condition for a solution of (3) to converge to 0 or 1 in $-\infty$ and $+\infty$. In the sequel, for $i = 1, 2$, we define the i-th moment of the kernel ϕ by

$$m_i := \int_{\mathbb{R}} |z|^i \phi(z)\, dz\,. \tag{7}$$

Lemma 3 (Sufficient condition for $u' \in L^2$). *Let $c \in \mathbb{R}$ and $u \in C_b^2(\mathbb{R})$ be a solution of (3). Assume $\mu\sqrt{m_2}\|u\|_{L^\infty} < |c|$. Then $u' \in L^2(\mathbb{R})$ and $\lim_{\pm\infty} u' = 0$.*

Proof. Let us define $M := \|u\|_{L^\infty}$ and $M' := \|u'\|_{L^\infty}$. Denote by W a potential associated with the underlying (local) monostable nonlinearity i.e. $W'(x) = x(1-x)$. We rewrite the equation as

$$cu' = -u'' - \mu u(1-u) - \mu u(u - \phi * u)\,,$$

multiply it by u', and then integrate from $-A < 0$ to $B > 0$ to get

$$c\int_{-A}^{B} {u'}^2 = \left[-\frac{1}{2}{u'}^2 - \mu W(u)\right]_{-A}^{B} - \mu \int_{-A}^{B} u'u(u - \phi * u)\,. \tag{8}$$

We denote by $I_{A,B}$ the last integral appearing above and use the Cauchy-Schwarz inequality to see

$${I_{A,B}}^2 \le \int_{-A}^{B} (u'u)^2 \int_{-A}^{B} (u - \phi * u)^2 \le M^2 \int_{-A}^{B} {u'}^2 \int_{-A}^{B} (u - \phi * u)^2\,. \tag{9}$$

Now, for a given x, we write

$$(u - \phi * u)(x) = \int_{\mathbb{R}} \phi(x-y)(u(x) - u(y))\, dy = \int_{\mathbb{R}} \int_0^1 \phi(x-y)(x-y)u'(x + t(y-x))\, dtdy\,,$$

so that another application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}(u - \phi * u)^2(x) \quad &\le \quad \int_{\mathbb{R}} \int_0^1 \phi(x-y)(x-y)^2\, dtdy \int_{\mathbb{R}} \int_0^1 \phi(x-y){u'}^2(x + t(y-x))\, dtdy \\ &\le \quad m_2 \int_0^1 \int_{\mathbb{R}} \phi(-z){u'}^2(x + tz)\, dzdt\,.\end{aligned}$$

Integrating this we discover

$$\int_{-A}^{B} (u - \phi * u)^2 \le m_2 \int_0^1 \int_{\mathbb{R}} \phi(-z) \int_{-A+tz}^{B+tz} {u'}^2(y)\, dydzdt\,.$$

Since $|u'| \leq M'$ we get, by cutting into three pieces,

$$\int_{-A+tz}^{B+tz} {u'}^2 \leq \int_{-A}^{B} {u'}^2 + 2t|z|M'^2 \,,$$

which in turn implies

$$\begin{aligned} \int_{-A}^{B} (u - \phi * u)^2 &\leq m_2 \int_{-A}^{B} {u'}^2 + 2m_2 M'^2 \int_0^1 \int_{\mathbb{R}} t\phi(-z)|z| \, dzdt \\ &\leq m_2 \int_{-A}^{B} {u'}^2 + m_2 m_1 M'^2 \,. \end{aligned} \tag{10}$$

If $R_{A,B} := \int_{-A}^{B} {u'}^2$, combining (8), (9) and (10) we see that

$$\begin{aligned} |c| R_{A,B} &\leq \left| \left[-\frac{1}{2} {u'}^2 - \mu W(u) \right]_{-A}^{B} \right| + \mu M \sqrt{R_{A,B}} \sqrt{m_2 R_{A,B} + m_2 m_1 M'^2} \\ &\leq M'^2 + 2\mu \|W\|_{L^\infty(-M,M)} + \mu \sqrt{m_2} M \sqrt{{R_{A,B}}^2 + m_1 M'^2 R_{A,B}} \,. \end{aligned}$$

If $\mu\sqrt{m_2}M < |c|$ then the upper estimate compels $R_{A,B} = \int_{-A}^{B} {u'}^2$ to remain bounded, so that $u' \in L^2$. Since u' is uniformly continuous on $\mathbb{R}$, this implies $\lim_{\pm\infty} u' = 0$. This concludes the proof of the lemma. □

Lemma 4 (Sufficient condition for $u(\pm\infty) \in \{0,1\}$). *Let $c \in \mathbb{R}$ and $u \in C_b^2(\mathbb{R})$ be a solution of (3). Assume*

$$\mu\sqrt{m_2}\|u\|_{L^\infty} < |c| \,. \tag{11}$$

Then $\lim_{+\infty} u$ and $\lim_{-\infty} u$ exist and belong to $\{0,1\}$.

Proof. Since the proof is similar on both sides we only work in $+\infty$. Denote by $\mathcal{A}$ the set of accumulation points of u in $+\infty$. Since u is bounded, $\mathcal{A}$ is not empty. Let $\theta \in \mathcal{A}$. There is $x_n \to +\infty$ such that $u(x_n) \to \theta$. Then $v_n(x) := u(x + x_n)$ solves

$$v_n'' + c v_n' = -\mu v_n (1 - \phi * v_n) \quad \text{ on } \mathbb{R} \,.$$

Since the L^∞ norm of the right hand side member is uniformly bounded with respect to n, the interior elliptic estimates imply that, for all $R > 0$, all $1 < p < \infty$, the sequence (v_n) is bounded in $W^{2,p}([-R,R])$. From Sobolev embedding theorem, one can extract $v_{\varphi(n)} \to v$ strongly in $C_{loc}^{1,\beta}(\mathbb{R})$ and weakly in $W_{loc}^{2,p}(\mathbb{R})$. It follows from Lemma 3 that

$$v'(x) = \lim_{n\to\infty} u'(x + x_{\varphi(n)}) = 0 \,.$$

Combining this with the fact that v solves

$$v'' + cv' = -\mu v(1 - \phi * v) \quad \text{ on } \mathbb{R} \,,$$

we collect $v \equiv 0$ or $v \equiv 1$. From $v(0) = \lim_n u(x_{\varphi(n)}) = \theta$ we deduce that $\theta \in \{0,1\}$. Since u is continuous, $\mathcal{A}$ is connected and therefore $\mathcal{A} = \{0\}$ or $\mathcal{A} = \{1\}$. Therefore $u(+\infty)$ exists and is equal to 0 or 1. □

3. Conclusion

3.1. Proof of Theorem 2

Let us consider a travelling wave (c,u) as in Lemma 1. In view of $\liminf_{-\infty} u > 0$ and Lemma 4, for $u(-\infty)=1$ to hold it is enough to have (11). Therefore we need to investigate further the bound $\|u\|_{L^\infty}$. To construct (c,u), the authors in [2] first consider the problem in a finite box $(-a,a)$. They prove *a priori* bounds for solutions in the box, use a Leray-Schauder degree argument to construct a solution (c_a,u_a) in the box and then pass to the limit $a\to\infty$ to construct (c,u) a solution on the line $\mathbb{R}$. One of the crucial *a priori* estimate is the existence of a constant

$$K_0 = K_0(\phi,\mu) := \left(\int_{\mathbb{R}} \phi(z)(1-\mu\frac{z^2}{2})_+\,dz\right)^{-1}$$

such that $\|u\|_{L^\infty} \le K_0$ (see [2, Lemma 3.1 and Lemma 3.10]). Hence, any wave with speed $c > \overline{c} = \mu\sqrt{m_2}K_0$ will satisfy (11). This completes the proof of Theorem 2. □

3.2. Comments on the bistable case

Let us conclude with a few comments concerning the bistable case. The local equation is given by $\partial_t u = \partial_{xx}u + u(u-\alpha)(1-u)$, where $0<\alpha<1$. It is well known that there is a unique (up to translation) monotonic travelling wave with the expected boundary conditions (4).

A difficult issue is now to search for (nontrivial) travelling waves solutions supported by the integro-differential equation

$$\partial_t u = \partial_{xx}u + u(u-\alpha)(1-\phi * u) \qquad x\in\mathbb{R}\,,\quad t>0\,, \tag{12}$$

that is (c,u) such that

$$-cu' = u'' + u(u-\alpha)(1-\phi*u) \qquad x\in\mathbb{R}\,, \tag{13}$$

supplemented with *ad hoc* boundary conditions. As far as we know, no result exists for such waves. For instance, among other things, nonlinearities such as $u(\phi*u-\alpha)(1-u)$ are treated in [9], but equation (12) does not fall into [9, equation (1.6)]. Indeed $g(u,v) = u(u-\alpha)(1-v)$ does not satisfy [9, hypotheses (H1)–(H2)] (which would ensure the stability of $u\equiv 0$ and $\equiv 1$). Moreover, the standard construction scheme used in [2] for the nonlocal Fisher-KPP equation cannot be applied straightforwardly to the bistable case. More precisely, proceeding as in [2], one can construct an approximated solution (c_a,u_a) defined on a bounded box $(-a,a)$ and then try to pass to the limit as $a\to\infty$. However, the change of sign of the nonlinearity around α generates some difficulties in the establishment of sharp *a priori* estimates on (c_a,u_a) and makes very delicate the comprehension of the behavior of a solution u in $\pm\infty$. In particular, the Harnack type arguments crucially used in [2] fail in this situation.

Nevertheless, even if the construction of travelling waves for the bistable case is still to be addressed, the L^2 estimates of Section 2 turn out to be of independent interest: it is straightforward to derive the following analogous of Lemma 4 for the bistable equation (13).

Lemma 5 (Sufficient condition for $u(\pm\infty) \in \{0, \alpha, 1\}$, bistable case). *Let $c \in \mathbb{R}$ and $u \in C_b^2(\mathbb{R})$ be a solution of (13). Assume*

$$\sqrt{m_2}\|u\|_{L^\infty}^2 < |c| \,. \tag{14}$$

Then $\lim_{+\infty} u$ and $\lim_{-\infty} u$ exist and belong to $\{0, \alpha, 1\}$.

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