Propagation phenomena in monostable integro-differential equations: acceleration or not?

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Contents

Abstract

We consider the homogeneous integro-differential equation $\partial_t u = J * u - u + f(u)$ with a monostable nonlinearity f. Our interest is twofold: we investigate the existence/non existence of travelling waves, and the propagation properties of the Cauchy problem.

When the dispersion kernel J is exponentially bounded, travelling waves are known to exist and solutions of the Cauchy problem typically propagate at a constant speed [22], [26], [7], [11], [10], [27]. On the other hand, when the dispersion kernel J has heavy tails and the nonlinearity f is non degenerate, i.e $f'(0) > 0$, travelling waves do not exist and solutions of the Cauchy problem propagate by accelerating [20], [27], [14]. For a general monostable nonlinearity, a dichotomy between these two types of propagation behavior is still not known.

The originality of our work is to provide such dichotomy by studying the interplay between the tails of the dispersion kernel and the Allee effect induced by the degeneracy of f, i.e. $f'(0) = 0$. First, for algebraic decaying kernels, we prove the exact separation between existence and non existence of travelling waves. This in turn provides the exact separation between non acceleration and acceleration in the Cauchy problem. In the latter case, we provide a first estimate of the position of the level sets of the solution.

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1 Introduction

In this work, we are interested in the propagation phenomena for solutions $u(t,x)$ of homogeneous integro-differential equations of the form

$$\partial_t u = J * u - u + f(u), \quad t > 0,\, x \in \mathbb{R}. \tag{1}$$

In population dynamics models, u stands for a population density, the nonlinearity f encodes the demographic assumptions and J is a nonnegative dispersal kernel of total mass 1, allowing to take into account long distance dispersal events. Here, we consider nonlinearities f of the monostable type, namely $f(0) = f(1) = 0$ and $f > 0$ on $(0,1)$. Precise assumptions on J (heavy tails) and f (degeneracy at 0) will be given later one.

When f is non degenerate at 0, that is $f'(0) > 0$, it is known that the equation (1) exhibits some propagation phenomena: starting with some nonnegative nontrivial compactly supported initial data, the corresponding solution $u(t,x)$ converges to 1, its stable steady state, at large time and locally uniformly in space. This is referred as the *hair trigger effect* [4]. Moreover, in many cases, the convergence to 1 can be precisely characterised. For example, when f is a KPP nonlinearity —meaning $f(s) \le f'(0)s$ for all $s \in (0,1)$— and J is exponentially bounded, that is

$$\exists \lambda > 0, \quad \int_{\mathbb{R}} J(z) e^{\lambda |z|}\, dz < +\infty, \tag{2}$$

equation (1) admits travelling waves whose minimal speed c^* completely characterises the convergence $u(t,x) \to 1$, see [22], [26], [7], [11], [10].

For non degenerate monostable nonlinearities f, when the condition (2) is relaxed, allowing dispersion kernels with *heavy tails*, a new propagation phenomena appears: *acceleration*. This phenomenon for equation (1) was first heuristically obtained by Medlock and Kot [20] and mathematically described in [27], [14]: Yagisita [27] proves the non existence of travelling waves, and Garnier [14] studies the acceleration in the Cauchy problem.

Remark 1.1. *Acceleration phenomena for positive solutions of a Cauchy problem also appear in other contexts ranging from standard reaction diffusion equations [16], [1], to homogeneous equations involving fractional operators [6]. Let us also mention that acceleration phenomenon also appears in some porous media equations [18], [23].*

To capture this acceleration phenomenon, a precise description of the behavior of the level sets of $u(t,x)$ is required. More precisely, for $\lambda \in (0,1)$, let $E_\lambda(t)$ denote the set

$$E_\lambda(t) := \{x \in \mathbb{R} : u(t,x) = \lambda\}.$$

Then the acceleration can be characterised through the properties of $x^{\pm}_\lambda(t)$ representing the "largest" and the "smallest" element of $E_\lambda(t)$, i.e $x^+_\lambda(t) = \sup E_\lambda(t)$ and $x^-_\lambda(t) = \inf E_\lambda(t)$.

For example, when f is a KPP nonlinearity and $J(z) \sim \frac{C}{|z|^\alpha}$ $(\alpha > 2)$ for large z, the results of Garnier [14] assert that, for a solution of the Cauchy problem (1) with a nonnegative compactly supported initial data, the points $x^{\pm}_\lambda(t)$ move exponentially fast at large time: for any $\lambda \in (0,1)$ and $\varepsilon > 0$, there exists $\rho > f'(0)$ and $T_{\lambda,\varepsilon} > 0$ such that

$$e^{\frac{(f'(0)-\varepsilon)}{\alpha} t} \le |x^{\pm}_\lambda(t)| \le e^{\frac{\rho}{\alpha} t}, \ \forall t \ge T_{\lambda,\varepsilon}.$$

Similarly, Garnier [14] shows that, when f is a KPP nonlinearity and $J(z) \sim Ce^{-\beta|z|^\alpha}$ ($0 < \alpha < 1$, $\beta > 0$) for large z, the points $x_\lambda^\pm(t)$ move algebraically fast at large time: for any $\lambda \in (0,1)$ and $\varepsilon > 0$, there exists $\rho > f'(0)$ and $T_{\lambda,\varepsilon} > 0$ such that

$$\left(\frac{f'(0)-\varepsilon}{\beta}\right)^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}} \leq |x_\lambda^\pm(t)| \leq \left(\frac{\rho}{\beta}\right)^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}}, \; \forall t \geq T_{\lambda,\varepsilon}.$$

Note that the lower and upper bounds in the above estimates do not agree: tracking the level sets when acceleration occurs is a quite challenging task.

From a modelling point of view, it is natural to consider equation (1) with a monostable nonlinearity f which degenerates at 0, that is $f'(0) = 0$. This corresponds to assuming that the growth of the population at low density is not exponential any longer. In particular, this assumption induces an Allee effect on the evolution of the population, that is the maximal rate of production of new individuals is not achieved at low density, [3], [12], [25], [24], [19].

In this $f'(0) = 0$ degenerate case, the characterisation of the existence of travelling waves or acceleration in terms of the tails of J is far from understood. Indeed, under condition (2) for the kernel J, travelling fronts are known to exist [11], [10], and the Cauchy problem typically does not lead to acceleration [29]. But, when condition (2) is relaxed, the competition between heavy tails and the Allee effect has not yet been studied.

The purpose of this work is therefore to fill the above gap in the comprehension of propagation phenomena for equation (1), thus completing the picture describing the dichotomy between the existence of an accelerated propagation or not. First, in the spirit of [15] for a fractional version of (1), we derive an exact relation between the algebraic tails of the kernel J and the behavior of f near zero, which allows or not the existence of travelling waves. Then, in the spirit of [1] for a reaction diffusion equation with Allee effect and initial datum having heavy tails, we investigate the propagation phenomenon occurring in the Cauchy problem (1) with front like initial data. As a consequence of the travelling waves analysis, we derive the exact separation between non acceleration and acceleration. In the latter case, we give some estimates on the "speed" of expansion of the level-sets of the solution.

2 Assumptions and main results

Before stating our results, let us first present our assumptions on the dispersal kernel J and the degenerate monostable nonlinearity f.

Assumption 2.1 (Dispersal kernel for existence of waves)**.** $J : \mathbb{R} :\to [0,\infty)$ *is continuous, of total mass* $\int_{\mathbb{R}} J(x)dx = 1$. *We assume that there is* $C > 0$ *such that*

$$J(x) \leq \frac{C}{|x|^\mu}, \quad \forall x \leq -1, \quad \textit{for some } \mu > 2, \tag{3}$$

and

$$J(x) \leq \frac{C}{x^\alpha}, \quad \forall x \geq 1, \quad \textit{for some } \alpha > 2. \tag{4}$$

Symmetry is not assumed. As will be clear in the following, the important tail is the right one. In order to prove non existence we need to assume slightly more.

Assumption 2.2 (Dispersal kernel for non existence of waves)**.** *J satisfies Assumption 2.1 with* (4) *replaced by*

$$\frac{1/C}{x^\alpha} \leq J(x) \leq \frac{C}{x^\alpha}, \quad \forall x \geq 1, \quad \textit{for some } \alpha > 2. \tag{5}$$

Assumption 2.3 (Degenerate monostable nonlinearity)**.** *$f : [0,1] \to [0, \|f\|_\infty]$ is of the class C^1, and is of the monostable type, in the sense that*

$$f(0) = f(1) = 0, \quad f > 0 \quad \textit{on } (0,1).$$

The steady state 0 is degenerate, in the sense that

$$f(u) \sim ru^\beta, \ \textit{as } u \to 0, \quad \textit{for some } r > 0, \beta > 1, \tag{6}$$

whereas the steady state 1 is stable, in the sense that

$$f'(1) < 0. \tag{7}$$

The simplest example of a monostable nonlinearity involving such a degenerate Allee effect is given by $f(u) = ru^\beta(1-u)$.

Definition 2.4 (Travelling wave)**.** *A travelling wave for equation* (1) *is a couple (c,u) where $c \in \mathbb{R}$ is the speed, and u is a decreasing profile satisfying*

$$\begin{cases} J * u - u + cu' + f(u) = 0 & \textit{on } \mathbb{R}, \\ u(-\infty) = 1, \quad u(\infty) = 0. \end{cases}$$

Notice that if $c \neq 0$ then it follows from the equation that the profile u of a travelling wave has to be in $C^1_b(\mathbb{R})$. On the other hand, if $c = 0$, the situation is more tricky and, as observed in [10], it may happen that the above travelling wave problem admits infinitely many solutions that are not continuous.

Theorem 2.5 (Existence of travelling waves)**.** *Let Assumptions 2.1 and 2.3 hold. Assume*

$$\beta \geq 1 + \frac{1}{\alpha - 2}. \tag{8}$$

Then there is $c^* > J_1 := \int_{\mathbb{R}} yJ(y)dy$ such that for all $c \geq c^*$ equation (1) *admits travelling waves (c,u), whereas, for all $c < c^*$ equation* (1) *does not admit travelling wave.*

On the one hand, for any ("small") $\beta > 1$ (measuring the degeneracy of f in 0), one can find some (large) α (measuring the right tail of the kernel J) so that (1) supports the existence of travelling waves. On the other hand, for any ("small") $\alpha > 2$, one can find some (large) β so that (1) supports the existence of travelling waves.

Corollary 2.6 (Kernels lighter than algebraic)**.** *Let Assumption 2.1 hold, with* (4) *replaced by: for all $\alpha > 2$, there is $C_\alpha > 0$ such that*

$$J(x) \leq \frac{C_\alpha}{|x|^\alpha}, \quad \forall x \geq 1. \tag{9}$$

Let Assumption 2.3 hold. Then there is c^* such that for all $c \geq c^*$ equation (1) *admits travelling waves (c,u), whereas, for all $c < c^*$ equation* (1) *does not admit travelling wave.*

The above result is independent on $\beta > 1$ and is valid, among others, for kernels satisfying

$$J(x) \le Ce^{-a|x|/(\ln|x|)}, \quad \forall x \ge 2, \quad \text{for some } a > 0,$$

or

$$J(x) \le Ce^{-a|x|^b}, \quad \forall x \ge 1, \quad \text{for some } a > 0,\ 0 < b < 1,$$

for which travelling waves do not exist in the KPP case [14]. The proof is obvious: for a given $\beta > 1$, select a large $\alpha > 2$ such that (8) holds, and then combine (9) with Theorem 2.5 (more precisely the fact that the construction of an adequate supersolution is enough to prove the theorem, see Section 3).

Similarly, for strongly degenerate monostable f, we have the following consequence.

Corollary 2.7 (Strongly degenerate nonlinearity)**.** *Let Assumption 2.1 hold. Let Assumption 2.3 hold, with* (6) *replaced by: for all* $\beta > 1$*, there is* $C_\beta > 0$ *such that*

$$\lim_{u \to 0^+} \frac{f(u)}{u^\beta} \le C_\beta. \tag{10}$$

Then there is c^* *such that for all* $c \ge c^*$ *equation* (1) *admits travelling waves* (c, u)*, whereas, for all* $c < c^*$ *equation* (1) *does not admit travelling wave.*

The above result is independent on $\alpha > 2$ and is valid, among others, for the Zel'dovich nonlinearity [28], [17], that is

$$f(u) = re^{-\frac{1}{u}}(1-u), \quad \text{for some } r > 0.$$

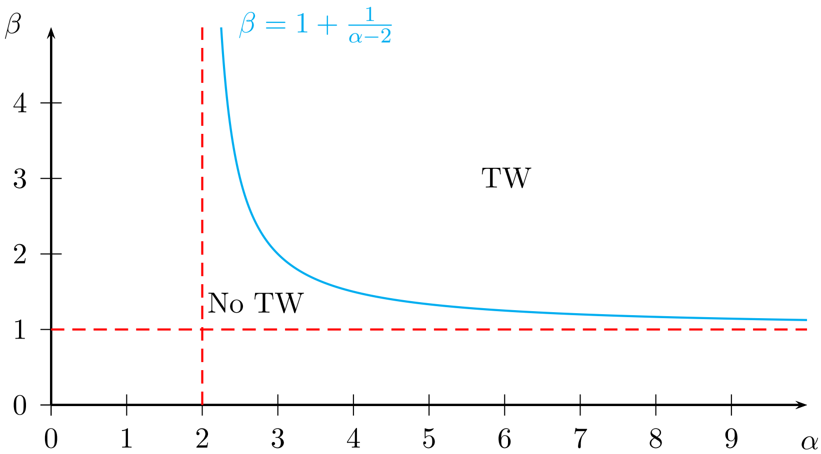
The proof is again obvious: for a given $\alpha > 2$, select a large $\beta > 1$ such that (8) holds, and then combine (10) with Theorem 2.5 to construct an adequate supersolution.

Next, we prove that the hyperbola separation (8) arising in Theorem 2.5 is optimal for the existence of travelling wave.

Theorem 2.8 (Non existence of travelling wave)**.** *Let Assumptions 2.2 and 2.3 hold. Assume*

$$\beta < 1 + \frac{1}{\alpha - 2}. \tag{11}$$

Then there is no travelling wave (c, u) *for equation* (1)*.*



We now turn to the Cauchy problem (1) with a front like initial datum u_0.

Assumption 2.9 (Front like initial datum)**.** *u_0 is of the class C^1, and satisfies*

(i) $0 \leq u_0(x) < 1, \quad \forall x \in \mathbb{R}$,

(ii) $\liminf_{x\to-\infty} u_0(x) > 0$,

(iii) $u \equiv 0$ *on* $[a, \infty)$ *for some* $a \in \mathbb{R}$.

Since f is Lipschitz and $0 \leq u_0 \leq 1$ the existence of a unique local solution $u(t, x)$ to the Cauchy problem (1) with initial datum u_0 is rather classical. Moreover, from the strong maximum priniciple, we know that $0 < u(t, \cdot) < 1$ as soon as $t > 0$ and the solution is global in time.

Theorem 2.5 and Theorem 2.8, are strong indications that, under assumption (8), assumption (11), no acceleration, respectively acceleration, should occur for the solution of the Cauchy problem. In order to clearly state such a result, for any $\lambda \in (0, 1)$ we define, in the spirit of the level sets used in [16], [14], [1], the (super) level sets of a solution $u(t, x)$ by

$$\Gamma_\lambda(t) := \{x \in \mathbb{R} : u(t, x) \geq \lambda\}.$$

Also we define the "largest" element of $\Gamma_\lambda(t)$ by

$$x_\lambda(t) := \sup \Gamma_\lambda(t) \in \mathbb{R} \cup \{-\infty, \infty\}.$$

Notice that, for compactly supported initial datum, it may happen that the solution get extinct at large time, which is referred as the *quenching phenomenon* [2], and thus $\Gamma_\lambda(t) = \emptyset$ at large time. This is one of our motivations for considering a front like initial datum. We can now state our first result on the Cauchy problem.

Proposition 2.10 (Acceleration or not in the Cauchy problem)**.** *Let Assumptions 2.2 and 2.3 hold. Let $u(t, x)$ be the solution of the Cauchy problem* (1) *with an initial datum u_0 satisfying Assumption 2.9.*

(i) Assume $\beta \geq 1 + \frac{1}{\alpha-2}$. Then there is $c_0 \in \mathbb{R}$ such that, for any $\lambda \in (0, 1)$,

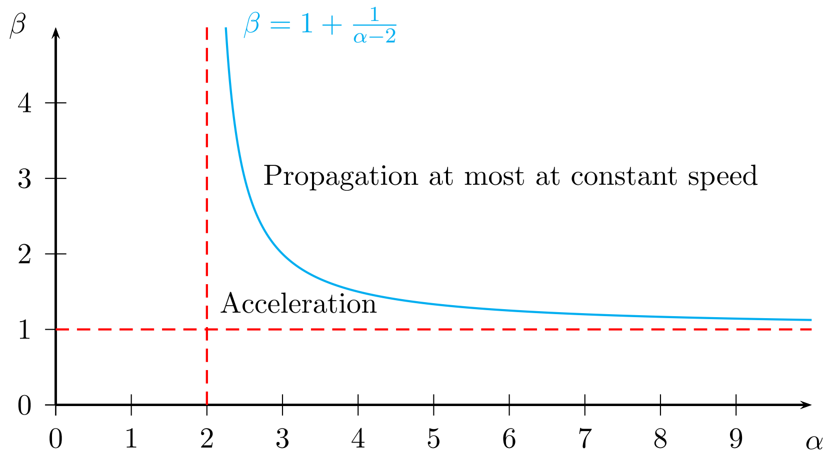
$$\limsup_{t\to\infty} \frac{x_\lambda(t)}{t} \leq c_0. \tag{12}$$

(ii) Assume $\beta < 1 + \frac{1}{\alpha-2}$. Then, for any $A \in \mathbb{R}$,

$$\lim_{t\to\infty} u(t, x) = 1 \qquad \textit{uniformly in } (-\infty, A], \tag{13}$$

and, for any $\lambda \in (0, 1)$,

$$\lim_{t\to\infty} \frac{x_\lambda(t)}{t} = +\infty. \tag{14}$$



In the first situation (i), we recover that the level sets of the solution $u(t,x)$ move to the right at most at a constant speed. Notice that the proof of (i) is rather standard (use the supersolution of Theorem 3.1 to control the propagation in the parabolic problem as in [1, Section 3]) and will be omitted. Notice also that assuming further that $\limsup_{x\to-\infty} u_0(x) < 1$, we can use the travelling wave with minimal speed as a supersolution, and thus replace c_0 by c^* in the conclusion (12). But, due to the lack of symmetry of the kernel J, it may happen that $c^* \leq 0$, see [10]. In such a case, we observe a propagation failure phenomenon for the solutions of the Cauchy problem.

On the other hand, the first part (13) of (ii) shows that invasion does occur (in particular $\Gamma_\lambda(t) \neq \emptyset$ at large time). Moreover the second pat (14) of (ii) indicates that the level sets of the solution move by accelerating.

Our last main result aim at precising the acceleration phenomenon (ii), by giving a first estimate of the actual position of $x_\lambda(t)$.

Theorem 2.11 (Further estimates on the acceleration phenomenon)**.** *Let Assumptions 2.2 and 2.3 hold. Let $u(t,x)$ be the solution of the Cauchy problem* (1) *with an initial datum u_0 satisfying Assumption 2.9. Assume that*

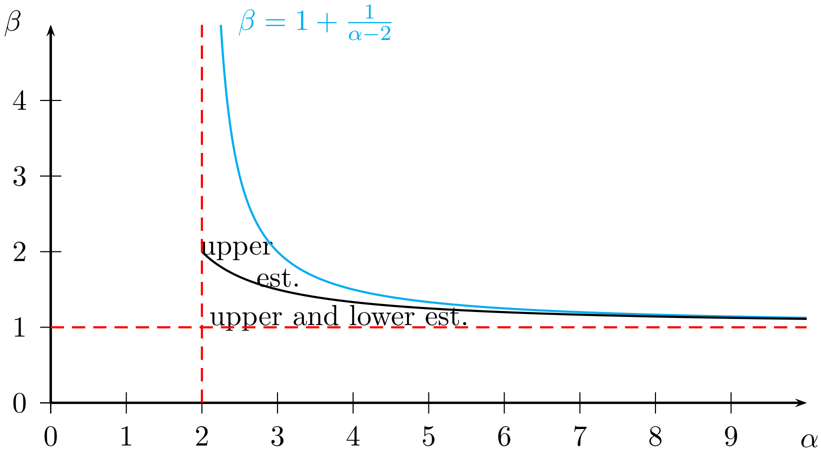
$$\beta < 1 + \frac{1}{\alpha - 2}. \tag{15}$$

Then there exists $\overline{C} > 0$ such that for all $\lambda \in (0,1)$, there is $T_\lambda > 0$ such that

$$x_\lambda(t) \leq \overline{C} t^{\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1}}, \quad \forall t \geq T_\lambda. \tag{16}$$

Moreover, under the stronger assumption $\beta < 1 + \frac{1}{\alpha-1}$, there exists $\underline{C} > 0$ such that for all $\lambda \in (0,1)$, there is $T'_\lambda > 0$ such that

$$\underline{C} t^{\frac{1}{(\alpha-1)(\beta-1)}} \leq x_\lambda(t) \leq \overline{C} t^{\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1}}, \quad \forall t \geq T'_\lambda. \tag{17}$$



Let us comment on the different exponents in the lower and upper estimate of (17), which is valid under assumption $\beta < 1 + \frac{1}{\alpha-1}$. We conjecture that the correct exponent is $\frac{1}{(\alpha-1)(\beta-1)} + \frac{1}{\alpha-1}$. To rigorously obtain the correct exponent a deeper analysis of the propagation phenomenon is needed, but this seems very involved. Indeed, the degeneracy of f near zero induces a possible quenching phenomenon for the Cauchy problem. This possibility is well known for classical reaction diffusion equations [4], [30], depends on β which measures the degeneracy of f at 0, and is very related to the so called *Fujita exponent* [13] for equation $\partial_t u = \Delta u + u^{1+p}$, $p > 0$. Very recently, the Fujita exponent was identified for the integro-differential equation $\partial_t u = J * u - u + u^{1+p}$, and the quenching phenomenon for (1) was analyzed [2]. This analysis paves the way to further studies of the acceleration in the Cauchy problem.

The paper is organized as follows. In Section 3, we prove existence of travelling waves in the regime (8), that is we prove Theorem 2.5. In Section 4, we prove non existence of travelling waves in the regime (11), that is we prove Theorem 2.8. Last, in Section 5, we study the acceleration phenomenon in the Cauchy problem, proving Proposition 2.10 (ii) and Theorem 2.11.

3 Travelling waves

In this section, we consider the regime (8) and construct travelling waves, that is we prove Theorem 2.5. The main task is the contruction of a supersolution as follows.

Theorem 3.1 (A supersolution)**.** *Let Assumptions 2.1 and 2.3 hold. Assume* (8)*. Then we can construct* $c_0 \in \mathbb{R}$ *and a decreasing* $w : \mathbb{R} \to (0,1)$ *satisfying* $w(x) = 1 - e^x$ *on* $(-\infty, -1)$*,* $w(x) = \frac{1}{x^{\alpha-2}}$ *on* (L, ∞)*, with* $L > 0$ *sufficiently large, and*

$$\varepsilon w'' + J * w - w + c_0 w' + f(w) \leq 0 \quad \text{on } \mathbb{R},$$

for any $0 \leq \varepsilon \leq 1$.

Proof. Define a smooth decreasing function w such that

$$w(x) := \begin{cases} 1 - e^x & \text{if } x \leq -1 \\ \dfrac{1}{x^p} & \text{if } x \geq L, \end{cases}$$

where $L > 1$ is chosen such that $1 - e^{-1} > \frac{1}{L^p}$. Since we want to show how the relation (8) appears, we let $p > 0$ free for the moment, and will chose $p = \alpha - 2$ only when it becomes necessary.

Notice that — in view of (6) and (7)— we can find some large $r > 0$ such that

$$f(w) \leq r w^\beta (1-w) =: g(w), \quad \forall w \in [0,1],$$

so it enough to prove $\varepsilon w'' + J * w - w + c_0 w' + g(w) \leq 0$ on $\mathbb{R}$.

Supersolution for $x >> 1$. Here, we work for $x \geq 2L$. Write

$$\begin{aligned} J * w(x) &= \int_{-\infty}^{-1} \frac{J(y)}{(x-y)^p} dy + \int_{-1}^{L} \frac{J(y)}{(x-y)^p} dy \\ &\quad + \int_{L}^{x-L} \frac{J(y)}{(x-y)^p} dy + \int_{x-L}^{\infty} J(y) w(x-y) dy =: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

In the sequel, $\varepsilon_0(z)$, $\varepsilon(x)$ denote functions which tend to 0 as $z \to 0$, $x \to \infty$ respectively, and which may change from place to place. We estimate below the terms I_k, $1 \leq k \leq 4$, as $x \to \infty$.

- We use the change of variable $z = y/x$ in I_1 and get

$$\begin{aligned} I_1 &= \frac{1}{x^{p-1}} \int_{-\infty}^{-1/x} \frac{J(xz)}{(1-z)^p} dz \\ &= \frac{1}{x^{p-1}} \int_{-\infty}^{-1/x} J(xz)(1 + pz(1 + \varepsilon_0(z))) dz, \end{aligned}$$

where we notice that $\varepsilon_0(z)$ remains bounded as $z \to -\infty$, and that $\varepsilon_0(z) \sim \frac{p+1}{2} z$ as $z \to 0$. In view of the control (3) of the left tail of the kernel J, we can therefore cut into three pieces, use the change of variable $y = xz$ in the first two terms, and get

$$\begin{aligned} I_1 &= \frac{1}{x^p} \int_{-\infty}^{-1} J(y) dy + \frac{p}{x^{p+1}} \int_{-\infty}^{-1} y J(y) dy + \frac{p}{x^{p-1}} \int_{-\infty}^{-1/x} J(xz) z \varepsilon_0(z) dz \\ &\leq \frac{1}{x^p} \int_{-\infty}^{-1} J(y) dy + \frac{p}{x^{p+1}} \int_{-\infty}^{-1} y J(y) dy + \frac{Cp}{x^{p+\mu-1}} \int_{-\infty}^{-1/x} \frac{1}{|z|^{\mu-1}} |\varepsilon_0(z)| dz. \end{aligned}$$

Notice that $\frac{1}{|\cdot|^{\mu-1}} \varepsilon_0(\cdot) \in L^1(-\infty, -1)$ since $\mu > 2$, and that $\frac{1}{|z|^{\mu-1}} |\varepsilon_0(z)| \sim \frac{p+1}{2} \frac{1}{|z|^{\mu-2}}$ as $z \to 0$. On the one hand, if $2 < \mu < 3$ then $\frac{1}{|\cdot|^{\mu-1}} \varepsilon_0(\cdot) \in L^1(-1, 0)$, so that

$$\int_{-\infty}^{-1/x} \frac{1}{|z|^{\mu-1}} |\varepsilon_0(z)| dz = \varepsilon(x) \frac{1}{x^{2-\mu}}$$

holds clearly. On the other hand, if $\mu \geq 3$ then $\frac{1}{|\cdot|^{\mu-1}} \varepsilon_0(\cdot) \notin L^1(-1, 0)$, so that

$$\int_{-\infty}^{-1/x} \frac{1}{|z|^{\mu-1}} |\varepsilon_0(z)| dz \sim \frac{p+1}{2} \int_{-\infty}^{-1/x} \frac{1}{|z|^{\mu-2}} dz = \varepsilon(x) \frac{1}{x^{2-\mu}},$$

as $x \to \infty$. In any case, we conclude that

$$I_1 \leq \frac{1}{x^p} \int_{-\infty}^{-1} J(y) dy + \frac{p}{x^{p+1}} \left[\int_{-\infty}^{-1} y J(y) dy + \varepsilon(x) \right]. \tag{18}$$

- For the term I_2, we use the same arguments to first obtain

$$I_2 \leq \frac{1}{x^p} \int_{-1}^{L} J(y)dy + \frac{p}{x^{p+1}} \int_{-1}^{L} yJ(y)dy + \frac{p}{x^{p-1}} \int_{-1/x}^{L/x} J(xz)z\varepsilon_0(z)dz.$$

Next, since

$$\int_{-1/x}^{L/x} J(xz)z\varepsilon_0(z)dz = \frac{1}{x^2} \int_{-1}^{L} J(y)\varepsilon_0\left(\frac{y}{x}\right) dy = \frac{1}{x^2}\varepsilon(x) \int_{-1}^{L} J(y)dy,$$

we conclude that

$$I_2 \leq \frac{1}{x^p} \int_{-1}^{L} J(y)dy + \frac{p}{x^{p+1}} \left[\int_{-1}^{L} yJ(y)dy + \varepsilon(x)\right]. \tag{19}$$

- We use the change of variable $z = y/x$ in I_3 and get

$$\begin{aligned} I_3 &= \frac{1}{x^{p-1}} \int_{L/x}^{1-L/x} \frac{J(xz)}{(1-z)^p} dz \\ &= \frac{1}{x^{p-1}} \left[\int_{L/x}^{1/2} \frac{J(xz)}{(1-z)^p} dz + \int_{1/2}^{1-L/x} \frac{J(xz)}{(1-z)^p} dz\right] \\ &= \frac{1}{x^{p-1}} \left[\int_{L/x}^{1/2} \frac{J(xz)}{(1-z)^p} dz + \int_{L/x}^{1/2} \frac{J(x(1-u))}{u^p} du\right]. \end{aligned} \tag{20}$$

Using the same arguments as above, the first term in the bracket above is recast as

$$\int_{L/x}^{1/2} \frac{J(xz)}{(1-z)^p} dz = \frac{1}{x} \int_{L}^{x/2} J(y)dy + \frac{p}{x^2} \left[\int_{L}^{x/2} yJ(y)dy + \int_{L}^{x/2} yJ(y)\varepsilon_0\left(\frac{y}{x}\right) dy\right],$$

where $\varepsilon_0(z)$ remains bounded as $z \to \infty$. Since $y \mapsto yJ(y) \in L^1(1,\infty)$ it follows from the dominated convergence theorem that $\int_L^{x/2} yJ(y)\varepsilon_0\left(\frac{y}{x}\right) dy \to 0$, as $x \to \infty$, so that

$$\int_{L/x}^{1/2} \frac{J(xz)}{(1-z)^p} dz = \frac{1}{x} \int_{L}^{x/2} J(y)dy + \frac{p}{x^2} \left[\int_{L}^{x/2} yJ(y)dy + \varepsilon(x)\right].$$

For the second term in (20) we use the control (4) of the right tail of the kernel J to collect

$$\begin{aligned} \int_{L/x}^{1/2} \frac{J(x(1-u))}{u^p} du &\leq \frac{C}{x^\alpha} \int_{L/x}^{1/2} \frac{1}{u^p(1-u)^\alpha} du \\ &\leq \frac{C2^\alpha}{x^\alpha} \int_{L/x}^{1/2} \frac{1}{u^p} du \\ &= \frac{C2^\alpha}{x^\alpha} \frac{1}{x^{2-\alpha}} \varepsilon(x) = \frac{1}{x^2}\varepsilon(x), \end{aligned}$$

if we further assume that $0 < p < \alpha - 1$. As a result, we collect

$$I_3 \leq \frac{1}{x^p} \int_{L}^{x/2} J(y)dy + \frac{p}{x^{p+1}} \left[\int_{L}^{x/2} yJ(y)dy + \varepsilon(x)\right]. \tag{21}$$

• For I_4 we use the crude estimate $w \le 1$ and the control (4) of the right tail of the kernel J to obtain

$$I_4 \le \int_{x-L}^{\infty} J(y)dy \le \int_{x-L}^{\infty} \frac{C}{y^\alpha}dy = \frac{C}{\alpha-1}\frac{1}{(x-L)^{\alpha-1}}. \tag{22}$$

• Summing (18), (19), (21) and (22) we arrive at

$$\begin{aligned} J*w(x) &\le \frac{1}{x^p}\int_{-\infty}^{x/2} J(y)dy + \frac{p}{x^{p+1}}\left[\int_{-\infty}^{x/2} yJ(y)dy + \varepsilon(x)\right] + \frac{C}{\alpha-1}\frac{1}{(x-L)^{\alpha-1}} \\ &\le \frac{1}{x^p} + \frac{p}{x^{p+1}}\left[J_1 + \varepsilon(x)\right] + \frac{C}{\alpha-1}\frac{1}{(x-L)^{\alpha-1}}, \end{aligned}$$

since $\int_{\mathbb{R}} J = 1$, and where $J_1 = \int_{\mathbb{R}} yJ(y)dy$. As a consequence we have, for any $0 \le \varepsilon \le 1$, (recall that $g(w) = rw^\beta(1-w) \le rw^\beta$)

$$\begin{aligned} &\varepsilon w''(x) + J*w(x) - w(x) + c_0 w'(x) + g(w(x)) \\ &\le \frac{p(p+1)}{x^{p+2}} - \frac{p}{x^{p+1}}(c_0 - J_1 + \varepsilon(x)) + \frac{C}{\alpha-1}\frac{1}{(x-L)^{\alpha-1}} + \frac{r}{x^{p\beta}} \\ &\le -\frac{p}{x^{p+1}}(c_0 - J_1 + \varepsilon(x)) + \frac{C}{\alpha-1}\frac{1}{(x-L)^{\alpha-1}} + \frac{r}{x^{p\beta}}. \end{aligned}$$

For the right-hand side member to be nonpositive for large positive x, one needs $p+1 \le \alpha-1$ and $p+1 \le p\beta$, that is $\frac{1}{\beta-1} \le p \le \alpha-2$. In view of assumption (8), such a choice is possible and the optimal one is $p = \alpha-2$, so that

$$\begin{aligned} &\varepsilon w''(x) + J*w(x) - w(x) + c_0 w'(x) + w^\beta(x) \\ &\le -\frac{\alpha-2}{x^{\alpha-1}}\left(c_0 - J_1 - \frac{C}{(\alpha-2)(\alpha-1)}\frac{1}{(1-\frac{L}{x})^{\alpha-1}} - \frac{r}{\alpha-2} + \varepsilon(x)\right). \end{aligned}$$

Choosing

$$c_0 > c_{right} := J_1 + \frac{C}{(\alpha-2)(\alpha-1)} + \frac{r}{\alpha-2},$$

we conclude that there is $M > 2$ large enough so that, for any $0 \le \varepsilon \le 1$,

$$\varepsilon w''(x) + J*w(x) - w(x) + c_0 w'(x) + g(w(x)) \le 0, \quad \forall x \ge M.$$

Supersolution for $x \le -1$. Here, we work for $x \le -1$. The non degeneracy of 1 makes the analysis easy "on the left". Using the crude estimate $J*w \le 1$, we get

$$\begin{aligned} &\varepsilon w''(x) + J*w(x) - w(x) + c_0 w'(x) + g((w(x)) \\ &\quad \le -\varepsilon e^x + 1 - (1-e^x) - c_0 e^x + r(1-e^x)^\beta e^x \\ &\quad \le -e^x(c_0 - 1 - r). \end{aligned}$$

Choosing $c_0 > c_{left} := 1 + r$, we get, for any $0 \le \varepsilon \le 1$,

$$\varepsilon w''(x) + J*w(x) - w(x) + c_0 w'(x) + g(w(x)) \le 0, \quad \forall x \le -1.$$

Supersolution everywhere. We now finalize our choices. For $p=\alpha-2$, we define $w(x)$ as above. Then we choose a speed

$$c_0 > \max\left(c_{right}, c_{left}, c_{middle} := \frac{(\max_{x\in[-1,M]} w''(x))^+ + 1 + r}{\min_{x\in[-1,M]} -w'(x)}\right).$$

It follows from the above computations that, for any $0 \le \varepsilon \le 1$,

$$\varepsilon w'' + J * w - w + c_0 w' + g(w) \le 0,$$

holds true in (M,∞), $(-\infty,-1)$ but also in $[-1,M]$ thanks to the crude estimates $\varepsilon \le 1$, $J*w-w\le 1$, $g(w)\le r$ and $c_0 > c_{middle}$. The theorem is proved. □

In view of [10, Theorem 1.3], the construction of a supersolution in Theorem 3.1 is enough to ensure the existence of travelling waves. More precisely, there is $c^* \le c_0$ such that for all $c \ge c^*$ equation (1) admits travelling waves (c,u), whereas, for all $c < c^*$ equation (1) does not admit travelling wave.

To complete the proof of Theorem 2.5, it remains to prove that $c^* > J_1$, which we do in the following a priori estimate on travelling wave. Notice that assumption (8) is not required in Lemma 3.2, so that its results remain valid in the regime (11) where there is no travelling wave, see Section 4.

Lemma 3.2 (Speed from below and integrability property)**.** *Let Assumptions 2.1 and 2.3 hold. Let (c,u) be a travelling wave. Then $c > J_1 = \int_{\mathbb{R}} yJ(y)dy$, and*

$$\int_0^{+\infty} (u(x))^\beta dx < +\infty. \tag{23}$$

Proof. We first claim (see below for a proof) that

$$I_R := \int_{-R}^{R} (J*u-u)(x)dx \to J_1, \qquad \text{as } R\to\infty. \tag{24}$$

Combining this with the travelling wave equation, $cu' \in L^1(\mathbb{R})$ and $f(u)\ge 0$ we get that $f(u)\in L^1(\mathbb{R})$, which in turn implies (23) since $f(u(x)) \sim ru^\beta(x)$ as $x\to\infty$. Also integrating the travelling wave equation on $\mathbb{R}$, we find

$$c - J_1 = \int_{\mathbb{R}} f(u(x))dx > 0,$$

and it only remains to prove the claim (24).

Let us first assume that the wave u is in $W^{1,\infty}(\mathbb{R})$ — which is in particular the case as soon as $c\neq 0$— so that one can write

$$I_R = \int_{-R}^{R}\int_{\mathbb{R}} J(y)(u(x-y)-u(x))dydx = \int_{-R}^{R}\int_{\mathbb{R}}\int_0^1 J(y)u'(x-ty)(-y)dtdydx.$$

The absolute value of the integrand is bounded by $\|u'\|_\infty |y| J(y)$ which belongs to $L^1(\mathbb{R})$, so that Fubini's theorem yields

$$I_R = -\int_{\mathbb{R}} yJ(y)\int_0^1\int_{-R}^{R} u'(x-ty)dxdtdy,$$

and thus

$$I_R = -\int_{\mathbb{R}} yJ(y) \int_0^1 (u(R-ty) - u(-R-ty))dtdy. \tag{25}$$

Now the boundary conditions $u(\infty) = 0$, $u(-\infty) = 1$ and the dominated convergence theorem yields (24).

It therefore only remains to consider the $c = 0$ case, for which we only know $u \in L^\infty(\mathbb{R})$. We use a mollifying argument. Let $(\rho_n)_{n\geq 0}$ be a sequence of mollifiers and define $u_n := \rho_n * u \in C^\infty(\mathbb{R})$, so that $\|u_n\|_\infty \leq \|u\|_\infty = 1$. Up to an extraction, $u_n \to u$ almost everywhere on $\mathbb{R}$, and, by the dominated convergence theorem, $J * u_n \to J * u$ on $\mathbb{R}$. Using again the dominated convergence theorem, we see that

$$I_R^n := \int_{-R}^{R} (J * u_n - u_n)(x)dx \to I_R, \quad \text{as } n \to \infty.$$

On the other hand, for a given $n \geq 0$, $u_n \in W^{1,\infty}(\mathbb{R})$ so that equality (25) applies to u_n and

$$\begin{aligned} I_R^n &= -\int_{\mathbb{R}} yJ(y) \int_0^1 (u_n(R-ty) - u_n(-R-ty))dtdy \\ &\to -\int_{\mathbb{R}} yJ(y) \int_0^1 (u(R-ty) - u(-R-ty))dtdy, \quad \text{as } n \to \infty. \end{aligned}$$

From the above $n \to \infty$ limits of I_R^n, we get that equality (25) is still true in this $c = 0$ case, and we conclude as above. □

4 Non existence of travelling wave

In this section, we consider the regime (11) and prove non existence of travelling wave, that is we prove Theorem 2.8.

We begin with an estimate of the nonlocal diffusion term for an algebraic tail which will then serve, twice, as a subsolution. This estimate is in the spirit of [21], where the nonlocal diffusion operator is the fractional Laplacian. Nevertheless, since our nonlocal diffusion operator does not share the homogeneity property (allowed by some singularity in zero) of the fractional Laplacian, we need to deal with an additional bad negative term in (26).

Lemma 4.1 (Estimate for an algebraic tail)**.** *Let Assumption 2.2 hold. For $p > 0$, let us define*

$$w(x) := \begin{cases} 1 & \text{if } x < 1 \\ \dfrac{1}{x^p} & \text{if } x \geq 1. \end{cases}$$

Then there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$J * w(x) - w(x) \geq \frac{C_1}{x^{\alpha-1}} - \frac{C_2}{x^{p+1}} + O\left(\frac{1}{x^{p+\alpha-1}}\right), \qquad \text{as } x \to +\infty. \tag{26}$$

Proof. For $x \geq 2$, we write

$$J * w(x) - w(x) = \int_{-\infty}^{x-1} J(y) \left(\frac{1}{(x-y)^p} - \frac{1}{x^p}\right) dy + \int_{x-1}^{+\infty} J(y)dy + \int_{x-1}^{+\infty} J(y)\frac{-1}{x^p}dy.$$

In view of (5), the second integral term above is larger than $\frac{C_1}{x^{\alpha-1}}$ for some $C_1 > 0$, whereas the third integral term is $O\left(\frac{1}{x^{p+\alpha-1}}\right)$ as $x \to +\infty$. In the first integral term, we perform the change of variable $y = xz$ and cut into three pieces to obtain

$$J * w(x) - w(x) = I_1 + I_2 + I_3 + \frac{C_1}{x^{\alpha-1}} + O\left(\frac{1}{x^{p+\alpha-1}}\right),$$

where

- $I_1 := \frac{1}{x^{p-1}} \int_{-\infty}^{-1/x} J(xz) \left(\frac{1}{(1-z)^p} - 1\right) dz$. In view of (3), we obtain

$$|I_1| \leq \frac{C}{x^{p+\mu-1}} \int_{-\infty}^{-1/x} \frac{1}{|z|^\mu} \left(1 - \frac{1}{(1-z)^p}\right) dz.$$

Since the above integrand is equivalent to $\frac{p}{|z|^{\mu-1}}$ as $z \to 0$ (with $\mu - 1 > 1$), we end up with $|I_1| = O\left(\frac{1}{x^{p+1}}\right)$ as $x \to +\infty$.

- $I_2 := \frac{1}{x^{p-1}} \int_{-1/x}^{1/x} J(xz) \left(\frac{1}{(1-z)^p} - 1\right) dz$. Hence

$$|I_2| \leq \frac{\|J\|_\infty}{x^{p-1}} \int_{-1/x}^{1/x} \left|\frac{1}{(1-z)^p} - 1\right| dz.$$

Since the above integrand is equivalent to $p|z|$ as $z \to 0$, we end up with $|I_2| = O\left(\frac{1}{x^{p+1}}\right)$ as $x \to +\infty$.

- $I_3 := \frac{1}{x^{p-1}} \int_{1/x}^{1-1/x} J(xz) \left(\frac{1}{(1-z)^p} - 1\right) dz \geq 0$.

Putting all together concludes the proof of the lemma. □

We can now prove below some a priori algebraic estimates of the tails of possible travelling waves.

Lemma 4.2 (A priori estimates of tails from below)**.** *Let Assumptions 2.2 and 2.3 hold. Let $\varepsilon > 0$ be given. Then for any travelling wave (c, u), there is a constant $K > 0$ such that*

$$u(x) \geq \frac{K}{x^{\alpha-2+\varepsilon}}, \text{ for all } x \geq 1.$$

Proof. For a travelling wave (c, u), since $f \geq 0$ on $[0, 1]$, we have

$$J * u - u + cu' \leq 0 \quad \text{on } \mathbb{R}. \tag{27}$$

On the other hand, Lemma 4.1 implies that, for $A > 1$ large enough, the function $w(x) := \mathbf{1}_{(-\infty,1]}(x) + \mathbf{1}_{(1,\infty)}(x) \frac{1}{x^{\alpha-2+\varepsilon}}$ satisfies

$$J * w - w + cw' > 0 \quad \text{on } (A, \infty). \tag{28}$$

Since $w \leq 1$ and $\inf_{(-\infty,A]} u > 0$, we can select $K > 0$ small enough so that $Kw(x) - u(x) < 0$ for all $x \in (-\infty, A]$. Assume by contradiction that there is $x_0 > A$ such that $Kw(x_0) - u(x_0) > 0$. Since $Kw(x) - u(x) \to 0$ as $x \to +\infty$, the function $Kw - u$ reaches some global maximum at some point $x_1 \in (A, \infty)$, which is contradicted by (27) and (28). This proves the lemma. □

The next lemma is of crucial importance for the proof of non existence of waves under assumption (11). Roughly speaking, if the tail of a travelling wave is rather heavy then it is actually very heavy. Notice that such a trick was also used in [15]. In contrast with the previous lemma, we shall need to keep a trace of the nonlinear term to improve the tail estimate.

Lemma 4.3 (Making the tail heavier)**.** *Let Assumptions 2.2 and 2.3 hold. Let (c,u) be a travelling wave. Assume that there are*

$$\frac{1}{\beta} < \gamma < \frac{1}{\beta - 1}, \tag{29}$$

and $K > 0$ such that

$$u(x) \geq \frac{K}{x^\gamma}, \text{ for all } x \geq 1. \tag{30}$$

Then, there is $M > 0$ such that

$$u(x) \geq \frac{M}{x^{\beta\gamma - 1}}, \text{ for all } x \geq 1.$$

Proof. Using $f(u) \sim ru^\beta$ as $u \to 0$ and estimate (30), we deduce that there is $\delta > 0$ such that $f(u(x)) \geq \frac{\delta}{x^{\beta\gamma}}$ if x is sufficiently large. Hence, we get the existence of $A > 1$ such that

$$J * u - u + cu' + \frac{\delta}{x^{\beta\gamma}} \leq 0 \text{ on } (A, \infty). \tag{31}$$

On the other hand, since $p := \beta\gamma - 1 > 0$, it follows from Lemma 4.1 that the function $w(x) := \mathbf{1}_{(-\infty,1]}(x) + \mathbf{1}_{[1,\infty)}(x)\frac{1}{x^{\beta\gamma-1}}$ satisfies

$$J * w(x) - w(x) + cw'(x) \geq \frac{C_1}{x^{\alpha-1}} - \frac{C_2'}{x^{\beta\gamma}} + O\left(\frac{1}{x^{\beta\gamma+\alpha-2}}\right),$$

as $x \to +\infty$, where $C_2' = C_2 + |c|(\beta\gamma - 1) > 0$.

Therefore, for any $0 < M < \frac{\delta/2}{C_2'}$, the function $z := Mw - u$ satisfies

$$J * z(x) - z(x) + cz'(x) \geq \frac{\delta/2}{x^{\beta\gamma}} + O\left(\frac{1}{x^{\beta\gamma+\alpha-2}}\right) > 0 \text{ on } (A, \infty), \tag{32}$$

up to enlarging $A > 1$ if necessary,

We conclude as in Lemma 4.2: we can select $0 < M < \frac{\delta/2}{C_2'}$ so that $z(x) = Mw(x) - u(x) < 0$ for all $x \in (-\infty, A]$. Assume by contradiction that there is $x_0 > A$ such that $z(x_0) > 0$. Since $z(x) = Mw(x) - u(x) \to 0$ as $x \to +\infty$, the function z reaches some global maximum at some point $x_1 \in (A, \infty)$, which is contradicted by (32). This proves the lemma. □

Equipped with the above a priori estimates and the integrability property (23) of Lemma 3.2, we can now prove Theorem 2.8.

Proof of Theorem 2.8. Let us assume by contradiction inequality (11) together with the existence of a travelling wave (c,u).

First case: $0 < \alpha - 2 < \frac{1}{\beta}$. In this regime, Lemma 4.2 and Lemma 3.2 are enough to derive a contradiction. Indeed, we select $\varepsilon > 0$ small enough so that $0 < \alpha - 2 + \varepsilon < \frac{1}{\beta}$. It follows from Lemma 4.2 that $u(x)^\beta \geq \frac{K^\beta}{x^{\beta(\alpha-2+\varepsilon)}}$, for all $x \geq 1$. Since $\beta(\alpha-2+\varepsilon) < 1$, this contradicts the integrability property (23).

Second case: $\frac{1}{\beta} \leq \alpha - 2 < \frac{1}{\beta-1}$. In this regime, we further need to iterate Lemma 4.3 to derive a contradiction. We first select $\varepsilon > 0$ small enough so that $\frac{1}{\beta} < \gamma := \alpha - 2 + \varepsilon < \frac{1}{\beta-1}$. It follows from Lemma 4.2 that the assumptions of Lemma 4.3 hold true, so that we can apply it once (at least). Notice that the recursive sequence

$$\gamma_0 = \gamma, \ \gamma_{n+1} = \beta\gamma_n - 1$$

tends to $-\infty$ as $n \to +\infty$, so that there is $N \geq 1$ such that

$$\gamma_N \leq \frac{1}{\beta} < \gamma_{N-1} < \cdots < \gamma_0 < \frac{1}{\beta - 1}.$$

This allows us to apply recursively Lemma 4.3 N times and to end up with a constant $D > 0$ such that $u(x) \geq \frac{D}{x^{\gamma_N}}$, for all $x \geq 1$. Since $\beta\gamma_N \leq 1$, this again contradicts the integrability property (23).

Theorem 2.8 is proved. $\square$

5 Acceleration in the Cauchy problem

Through this section we assume (11) and study the acceleration phenomenon arising in the Cauchy problem

$$\partial_t u(t,x) = J * u(t,x) - u(t,x) + f(u(t,x)) \quad t > 0, \ x \in \mathbb{R}, \tag{33}$$
$$u(0,x) = u_0(x) \quad x \in \mathbb{R}, \tag{34}$$

when u_0 is a front like initial data, in the sense of Assumption 2.9.

5.1 Proof of acceleration

Here we prove Proposition 2.10 (ii).

To do so, we need a preliminary result on ignition problems that serve as an approximation of our degenerate monostable problem. For $0 < \theta < 1$ we consider a smooth ignition nonlinearity $g_\theta : [0,1] \to \mathbb{R}$, meaning $g_\theta = 0$ on $[0,\theta] \cup \{1\}$, $g_\theta > 0$ on $(\theta, 1)$. As proved in [9], there is a unique speed $c_\theta \in \mathbb{R}$ and a unique decreasing profile U_θ solving the travelling wave problem

$$J * U_\theta - U_\theta + c_\theta U_\theta' + g_\theta(U_\theta) = 0 \quad \text{ on } \mathbb{R}, \tag{35}$$
$$U_\theta(-\infty) = 1, \quad U_\theta(0) = \theta, \quad U_\theta(\infty) = 0. \tag{36}$$

Notice that when $c_\theta \neq 0$, $U_\theta \in C^1$ and satisfies the equation in the classical sense. On the other hand, when $c_\theta = 0$, depending on g_θ the function U_θ may have some discontinuities. However, in such a situation (35) is satisfied almost everywhere and the limits and the normalisation (36) are still valid. As a consequence of the non existence of monostable waves Theorem 2.8, we can prove the following.

Proposition 5.1 (Speeds of a sequence of ignition waves)**.** *Let Assumptions 2.2 and 2.3 hold. Assume* (11)*. Let* $(g_n) = (g_{\theta_n})$ *be a sequence of ignition nonlinearities such that* $g_n \le g_{n+1} \le f$ *and* $g_n \to f$*. Let* (c_n, U_n) *be the associated sequence of travelling waves. Then*

$$\lim_{n\to\infty} c_n = +\infty.$$

Proof. Since $g_{n+1} \ge g_n$ it follows from [10, Corollary 2.3] that $c_{n+1} \ge c_n$. Assume by contradiction that $c_n \nearrow \bar{c}$ for some $\bar{c} \in \mathbb{R}$. We distinguish two cases.

Assume here $\bar{c} \neq 0$. There is thus an integer n_0 such that, for all $n \ge n_0$, $c_n \neq 0$. As a consequence, for all $n \ge n_0$, U_n is smooth and since any translation of U_n is a still a solution, without any loss of generality we can assume the normalisation $U_n(0) = 1/2$. Now, thanks to Helly's Theorem [5] and up to extraction, U_n converges to a monotone function $\bar{U}$ such that $\bar{U}(0) = \frac{1}{2}$. Also, from the equation and up to extraction, U_n also converges in $C^1_{loc}(\mathbb{R})$, and the limit has to be $\bar{U}'$. As a result, $\bar{U}$ is monotone and solves

$$\begin{cases} J * \bar{U} - \bar{U} + \bar{c}\bar{U}' + f(\bar{U}) = 0 \quad \text{on } \mathbb{R}, \\ \bar{U}(-\infty) = 1, \quad \bar{U}(0) = \frac{1}{2}, \quad \bar{U}(\infty) = 0. \end{cases}$$

In other words, we have contructed a monostable travelling wave under assumption (11), which is a contradiction with Theorem 2.8.

Assume here $\bar{c} = 0$. Since (c_n) is nondecreasing, either $c_n < 0$ for all n, either there is an integer n_0 such that $c_n = 0$ for all $n \ge n_0$. In the former case, since for all n U_n is smooth, without loss of generality U_n can be normalized by $U_n(0) = \frac{1}{2}$. We can then use the Helly's Theorem [5] and the normalisation to pass to the limit in the equation in a weak sense to obtain a monotone function $\bar{U}$ such that

$$\begin{cases} J * \bar{U} - \bar{U} + f(\bar{U}) = 0 \quad \text{almost everywhere in } \mathbb{R}, \\ \bar{U}(-\infty) = 1, \quad \bar{U}(0) = \frac{1}{2}, \quad \bar{U}(\infty) = 0, \end{cases}$$

which again contradicts Theorem 2.8. Let us now consider the remaining case, $c_n = 0$ for $n \ge n_0$. Observe that from Assumption 2.3 we can find $0 < s_0 < 1$ such that $s - f(s)$ is a one-to-one function in $[0, s_0]$ and, since $g_n \to f$ is of ignition type, $s - g_n(s)$ is also a one-to-one function in $[0, s_0]$ for all n. Now since for $n \ge n_0$, U_n satisfies $U_n - g_n(U_n) = J * U_n$, U_n has to be continuous in $[U_n^{-1}(s_0), \infty)$. Now, thanks to invariance by translation, we can assume that, for all $n \ge n_0$, $U_n(0) = s_0$. The sequence of monotone functions $(U_n)_{n\ge n_0}$ being bounded, thanks to Helly's Theorem [5] and the normalisation, as $n \to \infty$, U_n converges pointwise to a monotone function $\bar{U}$ solution of

$$\begin{cases} J * \bar{U} - \bar{U} + f(\bar{U}) = 0 \quad \text{on } \mathbb{R}, \\ \bar{U}(-\infty) = 1, \quad \bar{U}(0) = s_0, \quad \bar{U}(\infty) = 0, \end{cases}$$

which again contradicts Theorem 2.8. □

Remark 5.2. *Clearly, the results of Proposition 5.1 stand as well if we replace the ignition type nonlinearity* g_n *by a bistable type nonlinearity.*

We are now in the position to prove the first part (13) of Proposition 2.10 (*ii*).

Proof of (13). First, we prove (13) for the particular case where the initial datum u_0 is a smooth nonincreasing function such that

$$u_0(x) = \begin{cases} d_0 & \text{for } x \leq -1 \\ 0 & \text{for } x \geq 0, \end{cases} \tag{37}$$

for an arbitrary $0 < d_0 < 1$. Since u_0 is nonincreasing, we deduce from the comparison principle that, for all $t > 0$, the function $u(t, x)$ is still decreasing in x.

Let us now extend f by 0 outside the interval $[0, 1]$. From [9], Proposition 5.1 and Remark 5.2, there exists $0 < \theta < d_0$ and a Lipschitz bistable function $g \leq f$ — i.e. $g(0) = g(\theta) = g(1) = 0$, $g(s) < 0$ in $(0, \theta)$, $g(s) > 0$ in $(\theta, 1)$, and $g'(0) < 0$, $g'(1) < 0$, $g'(\theta) > 0$— such that there exists a smooth decreasing function U_θ and $c_\theta > 0$ verifying

$$\begin{aligned} &J * U_\theta - U_\theta + c_\theta U_\theta' + g(U_\theta) = 0 \quad \text{on } \mathbb{R}, \\ &U_\theta(-\infty) = 1, \qquad U_\theta(\infty) = 0. \end{aligned}$$

Let us now consider $v(t, x)$ the solution of the Cauchy problem

$$\begin{aligned} &\partial_t v(t, x) = J * v(t, x) - v(t, x) + g(v(t, x)) \quad \text{for } t > 0, x \in \mathbb{R}, \\ &v(0, x) = u_0(x). \end{aligned}$$

Since $g \leq f$, v is a subsolution of the Cauchy problem (33)-(34) and by the comparison principle, $v(t, x) \leq u(t, x)$ for all $t > 0$ and $x \in \mathbb{R}$.

Now, thanks to the global asymptotic stability result [8, Theorem 3.1], since $d_0 > \theta$ we know that there exists $\xi \in \mathbb{R}, C_0 > 0$ and $\kappa > 0$ such that for all $t \geq 0$

$$\|v(t, \cdot) - U_\theta(\cdot - c_\theta t + \xi)\|_{L^\infty} \leq C_0 e^{-\kappa t}.$$

Therefore, we have for all $t > 0$ and $x \in \mathbb{R}$,

$$u(t, x) \geq v(t, x) \geq U_\theta(x - c_\theta t + \xi) - C_0 e^{-\kappa t}.$$

Since $c_\theta > 0$, by sending $t \to \infty$, we get $1 \geq \liminf_{t\to\infty} u(t, x) \geq \lim_{t\to\infty} U_\theta(x - c_\theta t + \xi) - C_0 e^{-\kappa t} = 1$. As a result, for all x, $1 \geq \limsup_{t\to\infty} u(t, x) \geq \liminf_{t\to\infty} u(t, x) = 1$, and therefore $u(t, x) \to 1$ as $t \to \infty$. Since $u(t, x)$ is nonincreasing in x, the convergence is uniform on any set $(-\infty, A]$. This concludes the proof of (13) for our particular initial datum.

For a generic initial data satisfying Assumption 2.9, we can always, up to a shift in space, construct a smooth nonincreasing $\tilde{u_0}$ satisfying (37) and $\tilde{u_0} \leq u_0$. Since the solution $\tilde{u}(t, x)$ of the Cauchy problem starting from $\tilde{u_0}$ satisfies (13), so does $u(t, x)$ thanks to the comparison principle. □

Remark 5.3. *Notice that the above proof only uses elementary arguments and holds as well for other types of reaction diffusion equations, as soon as a travelling front solution with a positive speed exists when the nonlinearity considered is replaced by any nonlinearity of ignition or bistable type. In particular, it applies to solutions of Cauchy problems where the operator $J * u - u$ is replaced by a fractional Laplacian $-(-\Delta)^s u$, $0 < s < 2$, or the standard diffusive operator Δu.*

The property (13) now guarantees that for any $\lambda \in (0,1)$, the super level set $\Gamma_\lambda(t)$ is never empty for large time. Let us now prove the second part (14) of Proposition 2.10 (*ii*).

Proof of (14). Let $\lambda \in (0,1)$ be given. As above, there is no loss of generality to assume that the initial datum is as in the beginning of the proof of (13), so that $u(t,x)$ is nonincreasing in x. From this and (13), either for each $t>0$ large enough $\Gamma_\lambda(t)$ is bounded from above and $\Gamma_\lambda(t) = (-\infty, x_\lambda(t)]$, or $\Gamma_\lambda(t_0) = (-\infty,\infty)$ for some $t_0>0$. In the latter situation, using the constant λ as a subsolution, we see that, for all $t \geq t_0$, $\Gamma_\lambda(t) = (-\infty,\infty)$ so that $x_\lambda(t) = +\infty$ and we are done. In the sequel, we assume that for large t, says $t \geq t_1$, $x_\lambda(t) \in \mathbb{R}$.

Let $g \leq f$ be a smooth function such that $g(0) = g\left(\frac{1+\lambda}{2}\right) = 0$, $g(s) = 0$ for $s \leq 0$, $g(s) \sim f(s)$ as $s \to 0$, and $g(s) > 0$ for $s \in (0, \frac{1+\lambda}{2})$. For $\theta < \frac{1-\lambda}{2}$, let us consider the ignition type nonlinearity $g_\theta(s) := g(s-\theta)$ and let θ small, say $\theta \leq \theta_0$, such that for all $\theta \leq \theta_0$ there exists a smooth decreasing function U_θ and $c_\theta > 0$ such that

$$\begin{aligned}
&J * U_\theta - U_\theta + c_\theta U_\theta' + g_\theta(U_\theta) = 0 \quad \text{ on } \mathbb{R},\\
&U_\theta(-\infty) = \frac{1+\lambda}{2} + \theta, \qquad U_\theta(0) = \theta, \qquad U_\theta(\infty) = 0.
\end{aligned}$$

Then by a straightforward computation, we see that $\underline{U}(t,x) := U_\theta(x - c_\theta t) - \theta$ is a subsolution to equation (1). Notice that $\underline{U}(t,x) < \frac{1+\lambda}{2} < 1$ and $\underline{U}(0,x) \leq 0$. Since $u(t,x)$ converges uniformly to 1 in the set $(-\infty, 0]$, there thus exists $t_2 > t_1$ such that $u(t_2,x) \geq \underline{U}(0,x)$. Hence, from the comparison principle, $u(t+t_2,x) \geq \underline{U}(t,x) = U_\theta(x-c_\theta t) - \theta$ for all $t>0$ and $x \in \mathbb{R}$. Denoting by y_θ the point where $U_\theta(y_\theta) = \lambda + \theta$, this in turn implies that $x_\lambda(t) \geq y_\theta + c_\theta(t-t_2)$ for all $t>0$. As a result, $\liminf_{t\to\infty} \frac{x_\lambda(t)}{t} \geq c_\theta$. The above argument being independent of $\theta \leq \theta_0$ we get, thanks to Proposition 5.1,

$$\liminf_{t\to\infty} \frac{x_\lambda(t)}{t} \geq \lim_{\theta\to 0} c_\theta = +\infty,$$

which concludes the proof. □

5.2 Upper bound on the speed of the super level sets

Here we prove the upper bound (16) of Theorem 2.11. To do so, we construct an adequate supersolution.

Construction of a supersolution. For $p > 0$ to be specified later, let us define

$$v_0(x) := \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{1}{x^p} & \text{if } x > 1. \end{cases} \tag{38}$$

For $\gamma > 0$ to be specified later, let $w(\cdot, x)$ denote the solution of the Cauchy problem

$$\frac{dw}{dt}(t,x) = \gamma w^\beta(t,x), \quad w(0,x) = v_0(x),$$

that is

$$w(t,x) = \frac{1}{\left(v_0^{1-\beta}(x) - \gamma(\beta-1)t\right)^{\frac{1}{\beta-1}}}.$$

Notice that $w(t,x)$ is not defined for all times. When $x \leq 1$, $w(t,x)$ is defined for $t \in [0, \frac{1}{\gamma(\beta-1)})$ whereas for $x > 1$, $w(t,x)$ is defined for $t \in \left[0, T(x) := \frac{x^{p(\beta-1)}}{\gamma(\beta-1)}\right)$. Let $x_0(t)$ be such that

$$x_0(t) := [1 + \gamma(\beta-1)t]^{\frac{1}{p(\beta-1)}} \geq 1, \tag{39}$$

so that $w(t, x_0(t)) = 1$ and $w(t,x) < 1$ whenever $x > x_0(t)$. Last, we define

$$m(t,x) := \begin{cases} 1 & \text{if } x \leq x_0(t) \\ w(t,x) & \text{if } x > x_0(t), \end{cases} \tag{40}$$

and show below that m is a supersolution of (33)–(34), provided $p > 0$ and $\gamma > 0$ are appropriately chosen.

If (t,x) is such that $x \leq x_0(t)$, we see that $\partial_t m(t,x) = f(m(t,x)) = 0$, and

$$\partial_t m(t,x) - (J * m(t,x) - m(t,x) + f(m(t,x))) \geq 0,$$

since $m \leq 1$ by construction. Hence, it remains to consider the (t,x) such that $t > 0$ and $x > x_0(t)$, which we consider below.

In view of Assumption 2.3, there is $r_0 > 0$ such that $f(u) \leq r_0 u^\beta$ for all $0 \leq u \leq 1$. By definition of $m(t,x)$, we have $\partial_t m(t,x) = \gamma w^\beta(t,x)$ and $f(m(t,x)) \leq r_0 w^\beta(t,x)$. Next, for $\gamma > \gamma_0 := r_0 + 1$, let us define

$$x_\gamma(t) := \left[(\gamma - r_0)^{\frac{\beta-1}{\beta}} + \gamma(\beta-1)t\right]^{\frac{1}{p(\beta-1)}} > 1, \tag{41}$$

so that $x_0(t) < x_\gamma(t)$ and $w(t,x) \geq \left(\frac{1}{\gamma - r_0}\right)^{\frac{1}{\beta}}$ for $x_0(t) < x \leq x_\gamma(t)$. Thus, for $t > 0$ and $x_0(t) < x \leq x_\gamma(t)$,

$$\begin{aligned} \partial_t m(t,x) - (J * m(t,x) - m(t,x)) - f(m(t,x)) &\geq (\gamma - r_0) w^\beta(t,x) - J * m(t,x), \\ &\geq (\gamma - r_0) w^\beta(t,x) - 1, \\ &\geq 0. \end{aligned}$$

Hence, it remains to consider the (t,x) such that $t > 0$ and $x > x_\gamma(t)$, which we consider below.

Let us estimate more precisely $J * m(t,x)$ in the region $x > x_\gamma(t)$. To simplify the presentation, let us introduce the notations $q := p(\beta-1)$ and $\sigma := \gamma(\beta-1)t$. Let $K > 1$ to be specified later. We write

$$J * m(t,x) = \underbrace{\int_{-\infty}^{\frac{x_0(t)-x}{K}} J(-z) m(t, x+z)\, dz}_{I_1} + \underbrace{\int_{\frac{x_0(t)-x}{K}}^{\infty} J(-z) \frac{1}{[(x+z)^q - \sigma]^{\frac{p}{q}}}\, dz}_{I_2}. \tag{42}$$

In view of (41), we can select $\gamma_1 = \gamma_1(K) > \gamma_0$ large enough so that, for all $\gamma \geq \gamma_1$, all $x > x_\gamma(t)$, we have $x_0(t) - x < -K$. Therefore, from $m \leq 1$ and (4), we get (in the sequel C denotes a generic positive constant that may change from place to place)

$$I_1 \leq \int_{-\infty}^{\frac{x_0(t)-x}{K}} J(-z)\, dz \leq \frac{C K^{\alpha-1}}{(x - x_0(t))^{\alpha-1}}.$$

By choosing $q<1$ and using the definition of $x_0(t)$ we see that

$$\frac{1}{(x-x_0(t))^{\alpha-1}} \le \frac{1}{(x^q-x_0^q(t))^{\frac{\alpha-1}{q}}} \le w^{\frac{\alpha-1}{p}}\left(t,(x^q-1)^{1/q}\right).$$

Using that $q<1$ and $w(t,\cdot)$ is a decreasing function in $(x_0(t),\infty)$ (this can be seen by computing $\partial_x w = v_0' v_0^{-\beta} w^\beta \le 0$), we have for $x>>1$, say $x>A_0>\frac{2}{q}+1$,

$$w^{\frac{\alpha-1}{p}}\left(t,(x^q-1)^{1/q}\right) \le w^{\frac{\alpha-1}{p}}\left(t,x-\frac{2}{q}\right).$$

Up to enlarging A_0, for $x\ge A_0$ we have $4x^{q-1}<\frac{1}{2}$ and $x^q-4x^{q-1}\le\left(x-\frac{2}{q}\right)^q\le x^q-x^{q-1}$. Then for such A_0, we see that, for $x\ge x_0(t)+A_0$,

$$\begin{aligned}\frac{w(t,x-\frac{2}{q})}{w(t,x)} = \left(\frac{1}{1-w^{\beta-1}(t,x)\left[x^q-\left(x-\frac{2}{q}\right)^q\right]}\right)^{\frac{p}{q}} &\le \left(\frac{1}{1-4x^{q-1}w^{\beta-1}(t,x)}\right)^{\frac{p}{q}},\\ &\le \left(1+\frac{4x^{q-1}w^{\beta-1}(t,x)}{1-4x^{q-1}w^{\beta-1}(t,x)}\right)^{\frac{p}{q}},\\ &\le 2^{\frac{1}{\beta-1}}.\end{aligned}$$

Therefore, for γ large enough, say $\gamma\ge\gamma_2(A_0)$, we have for all $t>0$, $x\ge x_\gamma(t)\ge\ x_0(t)+A_0$,

$$I_1 \le C_1 K^{\alpha-1} w^{\frac{\alpha-1}{p}}(t,x). \tag{43}$$

We now turn to I_2. Using the change of variable $u=\frac{z}{x}$ and rearranging the terms, we get

$$I_2 = xw(t,x)\int_{\frac{x_0(t)}{Kx}-\frac{1}{K}}^{\infty} J(-xu)\frac{1}{\left(\frac{[1+u]^q-1}{1-\frac{\sigma}{x^q}}+1\right)^{p/q}}\,du = xw(t,x)(I_3+I_4), \tag{44}$$

where

$$I_3 := \int_{-\frac{1}{x}}^{\infty} J(-xu)\frac{1}{\left(\frac{[1+u]^q-1}{1-\frac{\sigma}{x^q}}+1\right)^{p/q}}\,du, \quad I_4 := \int_{\frac{x_0(t)}{Kx}-\frac{1}{K}}^{-\frac{1}{x}} J(-xu)\frac{1}{\left(\frac{[1+u]^q-1}{1-\frac{\sigma}{x^q}}+1\right)^{p/q}}\,du,$$

which we estimate below.

For I_3, since $u\in\left[-\frac{1}{x},\infty\right)$, $q<1$ and $(1+u)^q$ is a monotone increasing function, by using the definition of $w(t,x)$ we have

$$\frac{1}{\left(\frac{[1+u]^q-1}{1-\frac{\sigma}{x^q}}+1\right)^{p/q}} \le \frac{1}{(1-w^{\beta-1}(t,x))^{p/q}}.$$

Now, we know that for $x>x_\gamma(t)$, $w(t,x)\le\left(\frac{1}{\gamma-r_0}\right)^{1/\beta}<1$ so that a Taylor expansion yields a constant $\bar{C}(q)>0$ such that

$$\frac{1}{(1-w^{\beta-1}(t,x))^{p/q}} \le 1+\frac{p}{q}(1+\bar{C}(q))w^{\beta-1}(t,x),$$

so that

$$I_3 \leq \frac{1}{x} \int_{-1}^{+\infty} J(-z)\,dz + \frac{p}{q}(1+\bar{C}(q)) \frac{w^{\beta-1}(t,x)}{x} \int_{-1}^{+\infty} J(-z)\,dz. \tag{45}$$

For I_4, use the following claim, whose proof is postponed.

Claim 5.4. *For $q<1$, there exists $K(q)>0$ such that for all $t>0$, all $K \geq K(q)$, all $x > x_0(t)$, all $u \in \left[-\frac{1}{K} + \frac{x_0(t)}{Kx}, -\frac{1}{x}\right]$, we have*

$$\frac{[1+u]^q - 1}{1 - \frac{\sigma}{x^q}} \geq -\frac{1}{2}.$$

For $q<1$, we select $K \geq K(q)$ and $\gamma \geq \max\{\gamma_0, \gamma_1(K), \gamma_2(A_0)\}$. From the above claim, we deduce from Taylor expansion of the fraction $\frac{1}{(1-z)^{\frac{p}{q}}}$, there exists a constant $\tilde{C}(q)$ such that

$$\frac{1}{\left(\frac{[1+u]^q-1}{1-\frac{\sigma}{x^q}} + 1\right)^{p/q}} \leq 1 + \frac{p}{q}(1+\tilde{C}(q)) x^q w^{\beta-1}(t,x)\,(1 - [1+u]^q)\,.$$

Since $q<1$, and $J(z)|z| \in L^1(\mathbb{R})$ it follows that

$$I_4 \leq \frac{1}{x} \int_{\frac{x_0(t)-x}{K}}^{-1} J(-z)\,dz + \frac{p}{q}(1+\tilde{C}(q)) \frac{w^{\beta-1}(t,x)}{x} \int_{\frac{x_0(t)-x}{K}}^{-1} J(-z)|z|^q\,dz. \tag{46}$$

Owing to (44), (45) and (46), we get that, for some constant $C_2(q)>0$,

$$I_2 \leq w(t,x) \int_{\frac{x_0(t)-x}{K}}^{+\infty} J(-z)\,dz + C(q) w^\beta(t,x) \leq w(t,x) + C_2(q) w^\beta(t,x), \tag{47}$$

since $\int_{\mathbb{R}} J = 1$. Now, from (42), (43) and (47), we get, for $t>0$ and $x > x_\gamma(t)$,

$$J * m(t,x) - m(t,x) \leq \tilde{C}_1(q) w^{\frac{\alpha-1}{p}}(t,x) + C_2(q) w^\beta(t,x) \tag{48}$$

where $\tilde{C}_1(q) = C_1 K^{\alpha-1}(q)$. As a result,

$$\partial_t m - (J*m - m) - f(m) \geq w^\beta \left(\gamma - r_0 - C_2(q) - \tilde{C}_1(q) w^{\frac{\alpha-1}{p} - \beta}\right).$$

We are now close to conclusion. To validate the above computations we need $q = p(\beta-1) < 1$, and in the above inequality we need the exponent $\frac{\alpha-1}{p} - \beta$ to be nonnegative. In view of (8), these two conditions reduce to $p \leq \frac{\alpha-1}{\beta}$, so that we make the optimal choice $p = \frac{\alpha-1}{\beta}$. For this choice of p, and thus of q, we now choose $\gamma \geq \gamma^* := \max\{\gamma_0, \gamma_1(K), \gamma_2(A_0), r_0 + C_2(q) + \tilde{C}_1(q)\}$, so that the right hand side of the above inequality is positive. This completes the construction of the supersolution. □

Equipped with the above supersolution, we can now prove (16).

Proof of (16). In view of Assumption 2.9 on the initial datum u_0, we can assume, up to a shift in space, that $u_0 \leq v_0 = m(0,\cdot)$ where v_0 is as (38). It therefore follows from the comparison principle that $u(t,x) \leq m(t,x)$, where $m(t,x)$ is the supersolution (40) with $p = \frac{\alpha-1}{\beta}$ and

$\gamma \geq \gamma^*$. Hence, for $\lambda \in (0,1)$, the super level set $\Gamma_\lambda(t)$ of u is included in that of m. The latter can be explicitly computed, and we deduce

$$x_\lambda(t) \leq \left[\left(\frac{1}{\lambda}\right)^{\beta-1} + \gamma(\beta-1)t\right]^{\frac{\beta}{(\alpha-1)(\beta-1)}} \leq [2\gamma(\beta-1)t]^{\frac{\beta}{(\alpha-1)(\beta-1)}},$$

for $t \geq T_\lambda$, with $T_\lambda > 0$ large enough. This concludes the proof of (16). □

To complete this subsection, it remains to prove Claim 5.4.

Proof of Claim 5.4. Clearly $-\frac{1}{K} \leq u \leq 0$ so that for K large enough, say $K \geq K_0$, we have $(1+u)^q \geq 1 + \frac{2}{q}u$. Hence

$$\frac{[1+u]^q - 1}{1 - \frac{\sigma}{x^q}} \geq \frac{2}{q}\frac{u}{1-\frac{\sigma}{x^q}} \geq \frac{2}{qK}\frac{x_0(t)-x}{x^{1-q}}\frac{1}{x^q-\sigma},$$

since $\frac{x_0(t)}{Kx} - \frac{1}{K} \leq u \leq -\frac{1}{x}$. Since $x > x_0(t)$ we can write $x = \theta x_0(t)$ for some $\theta \in (1,\infty)$. Plugging this in the right hand side of the above inequality, using $\sigma = x_0^q(t) - 1$, and rearranging the terms we achieve

$$\begin{aligned}
\frac{[1+u]^q - 1}{1 - \frac{\sigma}{x^q}} &\geq -\frac{2}{qK}\frac{\theta - 1}{\theta - \theta^{1-q} + \theta^{1-q}x_0^{-q}(t)} \\
&= -\frac{2}{qK}\frac{\theta-1}{\theta-\theta^{1-q}}\frac{\theta-\theta^{1-q}}{\theta-\theta^{1-q}+\theta^{1-q}x_0^{-q}(t)} \\
&\geq -\frac{2}{qK}\frac{\theta-1}{\theta-\theta^{1-q}} \\
&= -\frac{2}{qK}\frac{1-\frac{1}{\theta}}{1-\frac{1}{\theta^q}}.
\end{aligned} \tag{49}$$

Since there is $C(q) > 0$ such that $\frac{1-\frac{1}{\theta}}{1-\frac{1}{\theta^q}} \leq C(q)$ for all $\theta > 1$, we end up with

$$\frac{[1+u]^q - 1}{1 - \frac{\sigma}{x^q}} \geq -\frac{2}{qK}C(q).$$

The claim is then proved by taking $K \geq K(q) := \max\left\{K_0, \frac{4C(q)}{q}\right\}$. □

5.3 Lower bound on the speed of the super level sets

Here we prove the lower bound (17) of Theorem 2.11. To measure the acceleration in this context, we use a subsolution that fills the space with a superlinear speed. The construction of this subsolution is inspired by that in [14] for nonlocal diffusion but in a KPP situation, and that in [1] in a Allee effect situation but for local diffusion.

Step one. It consits in using diffusion to gain an algebraic tail at time $t = 1$.

By Assumption 2.9 on the initial datum u_0, we can construct a nonincreasing $\tilde{u}_0$ such that $\tilde{u}_0 \leq u_0$ and

$$\tilde{u}_0(x) = \begin{cases} c_0 & \text{for } x \leq -R_0 - 1 \\ 0 & \text{for } x \geq -R_0, \end{cases} \tag{50}$$

for some $0 < c_0 < 1$ and $R_0 > 0$. From the comparison principle, it is enough to prove (17) for $u(t,x)$ the solution of (1) starting from $\tilde{u}_0$.

Since f is nonnegative, the comparison principle also implies $u(t,x) \geq v(t,x)$ for all $t > 0$, $x \in \mathbb{R}$, where $v(t,x)$ is the solution of the linear problem

$$\begin{aligned} &\partial_t v(t,x) = J * v(t,x) - v(t,x), \quad t > 0,\, x \in \mathbb{R}, \\ &v(0,x) = \tilde{u}_0(x). \end{aligned}$$

Again from the comparison principle we get $v(t,x) \geq e^{-t}(\tilde{u}_0(x) + tJ * \tilde{u}_0(x))$, and thus $v(1,x) \geq e^{-1}\left(\tilde{u}_0(x) + J * \tilde{u}_0(x)\right)$. In particular, for $x > 0$ we have

$$v(1,x) \geq e^{-1} J * \tilde{u}_0(x) \geq e^{-1} c_0 \int_{R_0+1+x}^{\infty} J(z)\, dz \geq \frac{c_0/C}{\alpha - 1} \frac{1}{(R_0 + 1 + x)^{\alpha-1}},$$

where we have used the tail estimate (5). As a result, we can find a small enough $d > 0$ such that

$$u(1,x) \geq v(1,x) \geq v_0(x) := \begin{cases} d & \text{for } x \leq 1 \\ \frac{d}{x^{\alpha-1}} & \text{for } x \geq 1. \end{cases} \tag{51}$$

Hence, from the comparison principle and up to a shift in time, it is enough to prove (17) for $u(t,x)$ the solution of (1) starting from v_0, which we do below.

Step two. It consists in the construction of the subsolution.

Let us consider the function $g(s) := s(1 - Bs)$, with $B > \frac{1}{2d}$. We have $g(s) \leq 0$ for all $s \geq \frac{1}{B}$ and $g(s) \leq \frac{1}{4B} \leq d$ for all $s \geq 0$.

As in the previous subsection, let $w(\cdot, x)$ denote the solution of the Cauchy problem

$$\frac{dw}{dt}(t,x) = \gamma w^{\beta}(t,x), \quad w(0,x) = v_0(x),$$

that is

$$w(t,x) = \frac{1}{\left(v_0^{1-\beta}(x) - \gamma(\beta - 1)t\right)^{\frac{1}{\beta-1}}},$$

where v_0 is defined in (51). Notice that $w(t,x)$ is not defined for all times. When $x \leq 1$, $w(t,x)$ is defined for $t \in [0, \frac{1}{d^{\beta-1}\gamma(\beta-1)})$, whereas for $x > 1$, $w(t,x)$ is defined for $t \in \left[0, T(x) := \frac{x^{(\alpha-1)(\beta-1)}}{d^{\beta-1}\gamma(\beta-1)}\right)$. Let us define

$$x_B(t) := d^{\frac{1}{\alpha-1}} \left[B^{\beta-1} 2^{\beta-1} + \gamma(\beta - 1)t\right]^{\frac{1}{(\alpha-1)(\beta-1)}} > 1, \tag{52}$$

so that $w(t, x_B(t)) = \frac{1}{2B}$.

For $x > 1$ and $0 < t < T(x)$, we compute

$$\begin{aligned} &\partial_x w(t,x) = v_0'(x) v_0^{-\beta}(x) w^{\beta}(t,x) \leq 0, \\ &\partial_{xx} w(t,x) = v_0^{-\beta}(x) w^{\beta}(t,x) \left(v_0''(x) + \beta \frac{(v_0'(x))^2}{v_0(x)} \left[\left(\frac{w(t,x)}{v_0(x)} \right)^{\beta-1} - 1 \right] \right) \geq 0. \end{aligned}$$

Hence, for $t > 0$, $w(t, \cdot)$ is a decreasing convex function on at least $(x_B(t), \infty)$.

Let us now define

$$m(t,x) := \begin{cases} \frac{1}{4B} & \text{for } x \leq x_B(t) \\ g(w(t,x)) & \text{for } x > x_B(t). \end{cases}$$

Observe that: when $x > x_B(0)$, $m(0,x) = g(v_0(x)) \leq v_0(x)$; when $x < 1$, $m(0,x) = \frac{1}{4B} \leq d = v_0(x)$; when $1 \leq x \leq x_B(0)$, $v_0(x) \geq \frac{d}{x_B^{\alpha-1}(0)} = \frac{1}{2B}$ so that $m(0,x) \leq \frac{1}{4B} \leq v_0(x)$. Hence $m(0,x) \leq v_0(x)$ for all $x \in \mathbb{R}$. Let us now show that $m(t,x)$ a subsolution to (1) for an appropriate choice of γ and B.

First, notice that, since $g\left(\frac{1}{2B}\right) = \frac{1}{4B}$ and $g'\left(\frac{1}{2B}\right) = 0$, we see that $m \in C^1([0,\infty) \times \mathbb{R})$. We compute

$$\partial_t m(t,x) = \begin{cases} 0 & \text{for } x \leq x_B(t) \\ \gamma w^\beta(t,x)\,(1 - 2Bw(t,x)) & \text{for } x > x_B(t). \end{cases} \tag{53}$$

Also, since f satisfies (6) and (7), there exists a small $\delta > 0$ such that $f(u) \geq \delta u^\beta(1-u)$ for all $0 \leq u \leq 1$. As a result, we see that

$$f(m(t,x)) \geq \begin{cases} C_0 w^\beta(t, x_B(t)) & \text{for } x \leq x_B(t) \\ C_0 w^\beta(t,x) & \text{for } x > x_B(t), \end{cases} \tag{54}$$

where $C_0 := \frac{\delta}{2^\beta}\left(1 - \frac{1}{4B}\right)$. As far as the nonlocal diffusion term is concerned, thanks to the monotone behavior of m, we have

$$J * m(t,x) - m(t,x) \geq \int_x^{+\infty} J(x-y)(m(t,y) - m(t,x))\,dy =: \mathcal{I}(x),$$

and we estimate $\mathcal{I}(x)$ below.

Assume first $x \leq x_B(t)$, so that $m(t,x) = m(t, x_B(t))$ and by using the fundamental theorem of calculus

$$\begin{aligned} \mathcal{I}(x) &\geq \int_{x_B(t)}^{+\infty} J(x-y)(m(t,y) - m(t,x_B(t)))\,dy, \\ &= \int_{x_B(t)}^{+\infty} \int_0^1 J(x-y)(y - x_B(t))\partial_x m(t, x_B(t) + s(y - x_B(t)))\,dyds, \\ &= \int_0^{+\infty} \int_0^1 J(x - x_B(t) - z) z \partial_x m(t, x_B(t) + sz)\,dzds. \end{aligned}$$

Now, $w(t,\cdot)$ being a positive decreasing convex function in $(x_B(t), \infty)$, we have, for any $sz > 0$,

$$\partial_x m(t, x_B(t) + sz) = \partial_x w(t, x_B(t) + sz)\,(1 - 2Bw(t, x_B(t) + sz)) \geq \partial_x w(t, x_B(t)),$$

so that

$$J * m(t,x) - m(t,x) \geq \partial_x w(t, x_B(t)) \int_0^{+\infty} J(x - x_B(t) - z) z\,dz \geq \partial_x w(t, x_B(t)) \int_{\mathbb{R}} J(z)|z|\,dz.$$

As a result

$$J * m(t,x) - m(t,x) \geq C v_0'(x_B(t)) v_0^{-\beta}(x_B(t)) w^\beta(t, x_B(t)), \quad \forall x \leq x_B(t), \tag{55}$$

where $C := \int_{\mathbb{R}} J(z)|z|\,dz$.

Similarly, when $x > x_B(t)$ by using the fundamental theorem of calculus, we get

$$\mathcal{I}(x) = \int_0^{+\infty} \int_0^1 J(-z) z \partial_x w(t, x + sz) \left(1 - 2Bw(t, x + sz)\right) dz ds.$$

Then, by using the convexity and the monotonicity of $w(t, \cdot)$ in $(x_B(t), \infty)$, we achieve

$$J * m(t,x) - m(t,x) \geq C \partial_x w(t,x) = C v_0'(x) v_0^{-\beta}(x) w^{\beta}(t,x), \quad \forall x > x_B(t). \tag{56}$$

Collecting (53), (54), (55) and (56), we end up with

$$(\partial_t m - (J * m - m) - f(m))(t,x) \leq \begin{cases} -w^{\beta}(t, x_B(t)) \left[C_0 + h(t,x)\right] & \text{for } x \leq x_B(t) \\ -w^{\beta}(t,x) \left[C_0 + h(t,x) - \gamma\right] & \text{for } x > x_B(t), \end{cases}$$

where

$$h(t,x) = \begin{cases} C v_0'(x_B(t)) v_0^{-\beta}(x_B(t)) & \text{for } x \leq x_B(t) \\ C v_0'(x) v_0^{-\beta}(x) & \text{for } x > x_B(t). \end{cases}$$

We now choose $\gamma \leq \frac{C_0}{2}$. In view of the above inequalities, to complete the construction of the subsolution $m(t,x)$, it suffices to find a condition on B so that $h(t,x) \geq -\frac{C_0}{2}$ for all $t > 0$, $x \in \mathbb{R}$. From the definition of $h(t,x)$ and that of $v_0(x)$ in (51), this corresponds to achieve

$$x^{(\beta-1)(\alpha-1)-1} \leq \frac{C_0 d^{\beta-1}}{2C(\alpha-1)}, \quad \text{for all } t > 0, x \geq x_B(t).$$

Since $(\beta-1)(\alpha-1) < 1$, this reduces to the following condition on $x_B(0)$

$$x_B(0) \geq \left(\frac{C_0 d^{\beta-1}}{2C(\alpha-1)}\right)^{\frac{1}{1-(\beta-1)(\alpha-1)}}.$$

From (52) we have $x_B(0) = (2Bd)^{\frac{1}{\alpha-1}}$. Hence, in view of the definition of C_0, the above inequality holds by selecting $B \geq B_0$, with $B_0 > 0$ large enough. This concludes the construction of the subsolution $m(t,x)$.

Step three. It consits in using the subsolution to prove the lower estimate in (17).

Fix $\gamma > 0$ and $B_0 > 0$ as in the previous step so that $m(t,x)$ is a subsolution. From the comparison principle we get $m(t,x) \leq u(t,x)$, for all $t > 0$ and $x \in \mathbb{R}$. Recall that $m(t, x_{B_0}(t)) = \frac{1}{4B_0}$ and that $u(t,\cdot)$ is nonincreasing (since initial datum v_0 is) so that

$$u(t,x) \geq \frac{1}{4B_0}, \quad \forall x \leq x_{B_0}(t). \tag{57}$$

In particular, for any $0 < \lambda \leq \frac{1}{4B_0}$, the "largest" element $x_\lambda(t)$ of the super level set $\Gamma_\lambda(t)$ has to satisfy

$$x_\lambda(t) \geq x_{B_0}(t) \geq d^{\frac{1}{\alpha-1}} \left[\gamma(\beta-1)t\right]^{\frac{1}{(\alpha-1)(\beta-1)}},$$

which provides the lower estimate in (17).

It now remains to obtain a similar bound for a given $\frac{1}{4B_0} < \lambda < 1$. Let us denote by $w(t,x)$ the solution of (1) starting from a nonincreasing w_0 such that

$$w_0(x) = \begin{cases} \frac{1}{4B_0} & \text{if } x \leq -1 \\ 0 & \text{if } x \geq 0. \end{cases} \tag{58}$$

It follows from Proposition 2.10 (ii) that there is a time $\tau_\lambda > 0$ such that

$$w(\tau_\lambda, x) > \lambda, \quad \forall x \le 0. \tag{59}$$

On the other hand, it follows from (57) and the definition (58) that

$$u(T, x) \ge w_0(x - x_{B_0}(T)), \quad \forall T \ge 0, \forall x \in \mathbb{R},$$

so that the comparison principle yields

$$u(T + \tau, x) \ge w(\tau, x - x_{B_0}(T)), \quad \forall T \ge 0, \forall \tau \ge 0, \forall x \in \mathbb{R}.$$

In view of (59), this implies that

$$u(T + \tau_\lambda, x) > \lambda, \quad \forall T \ge 0, \forall x \le x_{B_0}(T).$$

Hence, for $t \ge \tau_\lambda$, the above implies

$$x_\lambda(t) \ge x_{B_0}(t - \tau_\lambda) = d^{\frac{1}{\alpha-1}} \left[B_0^{\beta-1} 2^{\beta-1} + \gamma(\beta-1)(t - \tau_\lambda)\right]^{\frac{1}{(\alpha-1)(\beta-1)}} \ge \underline{C} t^{\frac{1}{(\alpha-1)(\beta-1)}},$$

provided $t \ge T'_\lambda$, with $T'_\lambda > \tau_\lambda$ large enough. This concludes the proof of the lower estimate in (17). □

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