Populations facing a *nonlinear* environmental gradient: steady states and pulsating fronts

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Contents

Abstract

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We consider a population structured by a space variable and a phenotypical trait, submitted to dispersion, mutations, growth and nonlocal competition. This population is facing an *environmental gradient*: to survive at location x, an individual must have a trait close to some optimal trait $y_{opt}(x)$. Our main focus is to understand the effect of a *nonlinear* environmental gradient.

We thus consider a nonlocal parabolic equation for the distribution of the population, with $y_{opt}(x) = \varepsilon\theta(x)$, $0 < |\varepsilon| \ll 1$. We construct steady states solutions and, when θ is periodic, pulsating fronts. This requires the combination of rigorous perturbation techniques based on a careful application of the implicit function theorem in rather intricate function spaces. To deal with the phenotypic trait variable y we take advantage of a Hilbert basis of $L^2(\mathbb{R})$ made of eigenfunctions of an underlying Schrödinger operator, whereas to deal with the space variable x we use the Fourier series expansions.

Our mathematical analysis reveals, in particular, how both the steady states solutions and the fronts (speed and profile) are distorted by the nonlinear environmental gradient, which are important biological insights.

Key Words: structured population, nonlocal reaction-diffusion equation, steady states, pulsating fronts, perturbation techniques.

AMS Subject Classifications: 35K57, 45K05, 35B10, 92D15.

1 Introduction

This paper is concerned with the nonlocal parabolic equation

$$\partial_t u = \partial_{xx} u + \partial_{yy} u + u \left(1 - A^2 \left(y - y_{opt}(x)\right)^2 - \int_{\mathbb{R}} u(t,x,y')\, dy'\right), \quad t > 0, x \in \mathbb{R}, y \in \mathbb{R}, \tag{1}$$

with

$$y_{opt}(x) := \varepsilon\theta(x), \quad \theta \in C_b(\mathbb{R}), \tag{2}$$

which serves as a model in evolutionary biology. Here $u = u(t,x,y)$ denotes the distribution of a population which, at each time $t > 0$, is structured by a space variable $x \in \mathbb{R}$, and a phenotypic trait $y \in \mathbb{R}$. This population is submitted to spatial dispersion, mutations, growth and competition. The spatial dispersion and the mutations are modeled by diffusion operators, namely $\partial_{xx}u$ and $\partial_{yy}u$. The intrinsic *per capita growth rate* of the population depends on both the location x and the phenotypic trait y. It is modeled by the confining term $1 - A^2 \left(y - y_{opt}(x)\right)^2$, where $A > 0$ is a constant that measures the strength of the selection. This corresponds to a population living in an *environmental gradient*: to survive at location x, an individual must have a trait close to the optimal trait $y_{opt}(x) = \varepsilon\theta(x)$. Finally, we consider a logistic regulation of the population distribution that is local in the spatial variable x and nonlocal in the phenotypic trait y. In other words, we consider that there exists, at each location, an intra-specific competition which takes place with all individuals whatever their trait.

The main input of this work is to analyze the case of a *nonlinear* environmental gradient. To do so, we consider that the optimal trait is described by (2) with $0 < |\varepsilon| \ll 1$, which corresponds to a nonlinear perturbation of the linear case $\varepsilon = 0$. First, under some natural assumptions, we construct steady states solutions, shedding light on how Gaussian solutions (corresponding to $\varepsilon = 0$) are distorted by the nonlinear perturbation. Next, we consider the case of a periodic perturbation, $\theta \in C(\mathbb{R}/L\mathbb{Z})$ for some $L > 0$, for which we construct *pulsating fronts* with a semi infinite interval of admissible speeds.

In ecology, an *environmental gradient* refers to a gradual change in various factors in space that determine the favoured phenotypic traits. Environmental gradients can be related to factors such as altitude, temperature, and other environment characteristics. It is now well documented that invasive species need to evolve during their range expansion to adapt to local conditions [24], [36]. Such issues are highly relevant in the context of the global warming [21], [23], or of the evolution of resistance of bacteria to antibiotics [34], [9]. Theoretical models therefore need to incorporate evolutionary factors [28], [37], [34]. In this context, let us mention the so-called "cane toad equation" which has led to rich mathematical results [10], [16], [15], [18]. On the other hand, equations having the form of (1) were developed in [40], [43], [42], [39].

Before discussing propagation phenomena in (1), let us briefly recall that *traveling fronts* are particular solutions that consist of a constant profile connecting zero to "a non-trivial state" and shifting at a constant speed. This goes back to the seminal works [26], [38] on the Fisher-KPP equation

$$\partial_t u = \Delta u + u(1-u), \quad t > 0, x \in \mathbb{R}^N,$$

and, among so many others, [6, 7], [25]. The construction of such solutions is much harder when the equation does not enjoy the comparison principle. One then usually needs to use topological degree arguments and the identification of the "non-trivial state" is typically missing, see e.g. [14], [2], [32] on the nonlocal Fisher-KPP equation.

As far as the mathematical analysis of (1) is concerned, one has to deal with the fine interplay between the space variable x and the phenotypic trait y, the fact that the phenotypic space is unbounded, and the nonlocal competition term. Because of the latter, equation (1) does not enjoy the comparison principle and its analysis is quite involved since many techniques based on the comparison principle — such as some monotone iterative schemes or the sliding method — are unlikely to be used.

Despite of that, the linear environmental gradient case, namely

$$y_{opt}(x) = Bx, \quad \text{for some } B \in \mathbb{R}, \tag{3}$$

is now rather well understood. In this case, depending on the sign of an underlying principal eigenvalue [3], either the population gets extinct, or it is able to adapt progressively to uncrowded zones and invade the environment. When propagation occurs, known results are the following. First, the $B = 0$ case allows a separation of variables trick, from which a rather exhaustive analysis can be performed [13]. Roughly speaking, traveling fronts can be written in the form $\Gamma_0(y)U(x-ct)$, where $\Gamma_0(y)$ is an underlying *ground state* or *principal eigenfunction* and $U(z := x - ct)$ a Fisher-KPP traveling wave with speed c. This fact will be precised and exploited later in the present work. On the other hand, when $B \neq 0$, variables cannot be separated and careful estimates of the nonlocal competition term are required. Thanks to rather sharp *a priori* estimates, Harnack and Bernstein type refined inequalities, traveling fronts are constructed in [3] and the determinacy of the spreading speed in the associated Cauchy problem is obtained in [1]. Very recently, accelerating invasions induced by initial *heavy tails* of the population distribution — see [30] and [27] for related results in absence of evolution— have been analysed in [41].

Last, let us mention that the case of a moving optimum

$$y_{opt}(t,x) = B(x - c_s t), \quad \text{for some } B \in \mathbb{R}, c_s > 0,$$

is also analyzed in [1]. This case serves as a model to study, e.g., the effect of global warming on the survival and propagation of a species: the favorable areas are shifted by the climate change at a given speed $c_s > 0$. The outcome is that there is an identified critical climate speed $c_s^* \geq 0$ such that $c > c_s^*$ implies extinction, whereas $c_s < c_s^*$ implies survival and invasion.

Nevertheless, the case of nonlinear environmental gradients is of great importance for applications, for instance in the context of development of resistance of pathogens to antibiotics. In this respect, let us mention the experimental set up of [9] where, thanks to mutation, *E. coli* bacteria are able to cross a four feet long petri dish on which the antibiotic concentration sharply increases[3].

As far as we know no significant mathematical results exist for model (1) when the environmental gradient $y_{opt}(x)$ is nonlinear. The reasons are, at least, threefold. First of all, it is much harder, if possible, to relate the issue to a underlying eigenvalue problem. Second, it is expected that the population may survive while being blocked in a restricted zone (so that invasion does not occur). Last, if invasion occurs, tracking the propagation of the solution is far from obvious since, among others, geometrical effects (via curvature) may appear along the optimal curve $y = y_{opt}(x)$.

Thus, in order to understand the situation where the optimal trait no longer depends linearly on space, our strategy is to consider the case (2) with $0 < |\varepsilon| \ll 1$, which we see as a nonlinear perturbation of the case (3) with $B = 0$ studied in [13].

[3]see the striking movie at https://www.youtube.com/watch?v=yybsSqcB7mE

Our first goal is to construct steady states, which we denote $n = n^\varepsilon(x, y)$, to (1). To do so, we will rely on rigorous perturbation techniques based on the implicit function theorem. We will also take advantage of the orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions of the underlying operator

$$-\frac{d^2}{dy^2} - (1 - A^2y^2).$$

This requires to work in rather intricate function spaces. Besides this rigorous theoretical construction, asymptotic expansions combined with numerical explorations enable to capture the distortion of the steady state by the nonlinear perturbation of the environmental gradient.

Our second goal is to analyze the propagation phenomena arising from model (1). To do so, for θ being L-periodic, we construct *pulsating fronts*. These particular solutions were first introduced by [44] in a biological context, and by Xin [49, 48, 47] in the framework of flame propagation, as natural extensions, in the periodic framework, of the aforementioned traveling fronts. By definition, a pulsating front is a speed $c_\varepsilon \in \mathbb{R}$ and a profile $\tilde{u}^\varepsilon(z, x, y)$, that is L-periodic in the x variable, such that

$$u^\varepsilon(t, x, y) := \tilde{u}^\varepsilon(x - c_\varepsilon t, x, y)$$

solves equation (1) and such that, as $z \to \pm\infty$, $\tilde{u}^\varepsilon(z, x, y)$ connects zero to a "non-trivial periodic state", a natural candidate being the steady state $n^\varepsilon(x, y)$ constructed previously. Equivalently, a pulsating front is a solution connecting zero to a "non-trivial periodic state", and that satisfies the constraint

$$u^\varepsilon\left(t + \frac{L}{c_\varepsilon}, x, y\right) = u^\varepsilon(t, x - L, y), \quad \forall (t, x, y) \in \mathbb{R}^3.$$

As far as monostable pulsating fronts are concerned, we refer among others to the seminal works of Weinberger [46], Berestycki and Hamel [11]. Let us also mention [35], [12], [29], [31] for related results. In contrast with these results and as mentioned above, model (1) does not enjoy the comparison principle. In such a situation, construction of pulsating fronts in a Fisher-KPP situation was recently achieved in [4] (see [19], [33] for an ignition type nonlinearity and a different setting). Another inherent difficulty of the present situation is to deal with both variables x (space) and y (phenotypic trait). To do so, we will first use the orthonormal basis of $L^2(\mathbb{R})$ mentioned above to deal with y and then use the Fourier series expansions to deal with x. Again, this is combined with a careful use of rigorous perturbation techniques based on the implicit function theorem. As far as we know, such perturbation arguments to construct pulsating fronts are rather used in the ignition [8] or bistable cases [22]. Besides this rigorous theoretical construction, our analysis reveals how the speed and profile of the fronts are modified by the nonlinear perturbation of the environmental gradient, which are very relevant for biological applications.

2 Main results

Letting

$$r(y) := 1 - A^2y^2, \quad A > 0, \tag{4}$$

equation (1) is recast

$$\partial_t u = \partial_{xx} u + \partial_{yy} u + u\left(r\left(y - \varepsilon\theta(x)\right) - \int_{\mathbb{R}} u(t, x, y')\, dy'\right). \tag{5}$$

Remark 1 (Quadratic choice)**.** *If $r : \mathbb{R} \to \mathbb{R}$ is continuous and confining, that is $\lim_{|y|\to\infty} r(y) = -\infty$, then the operator $-\frac{d^2}{dy^2} - r(y)$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$, and has discrete spectrum. There exists an orthonormal basis $\{\Gamma_k\}_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$ consisting of eigenfunctions, namely*

$$-\Gamma_k'' - r(y)\Gamma_k = \lambda_k \Gamma_k, \quad \|\Gamma_k\|_{L^2} = 1,$$

with corresponding eigenvalues $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to +\infty$ of finite multiplicity. Assuming that the confinement is, say, polynomial we may handle such per capita growth rate r as in [5]. For *the clarity of the exposition (in particular some relations between the eigenfunctions are helpful, see subsection 3.3) we have nonetheless decided to consider the quadratic case (4) which, anyway, reveals all the possible features of the model.*

In the sequel, we denote by $(\lambda_0, \Gamma_0(y))$ the principal eigenelements of $-\frac{d^2}{dy^2} - r(y)$, namely

$$\begin{cases} -\Gamma_0'' - (1 - A^2y^2)\Gamma_0 = \lambda_0\Gamma_0 & \text{in } \mathbb{R}, \\ \Gamma_0 > 0 & \text{in } \mathbb{R}, \\ ||\Gamma_0||_{L^2} = 1, \end{cases}$$

that is

$$\lambda_0 = A - 1, \quad \Gamma_0(y) = \left(\frac{A}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}Ay^2}.$$

We first state that, as soon as $\lambda_0 > 0$ and $|\varepsilon| \ll 1$, extinction of the population occurs for rather general initial data, including in particular the case of continuous compactly supported ones.

Proposition 2 (Extinction)**.** *Assume $\lambda_0 > 0$. Let us fix $0 < \mu_0 < \lambda_0$ and $\mu_0 + 1 < a < A = \lambda_0 + 1$. Let $\theta \in C_b(\mathbb{R})$. Then there is $\varepsilon_0 > 0$ such that, for any $|\varepsilon| < \varepsilon_0$, the following holds: any global nonnegative solution $u = u^\varepsilon(t, x, y)$ of (5), starting from a initial data $u_0 = u_0(x, y)$ such that*

$$M = M(u_0) := \sup_{(x,y)\in\mathbb{R}^2} u_0(x, y)e^{\frac{1}{2}ay^2} < +\infty, \tag{6}$$

goes extinct exponentially fast as $t \to +\infty$. More precisely, we have

$$0 \le u(t, x, y) \le Me^{-\mu_0 t}e^{-\frac{1}{2}ay^2}, \quad \textit{for all } t \ge 0, x \in \mathbb{R}, y \in \mathbb{R}.$$

When $\lambda_0 \ge 0$, extinction in the linear case $\varepsilon = 0$ is easily proved thanks to the comparison principle since the nonlocal term is "harmless" when searching an estimate *from above*. Hence, when $\lambda_0 > 0$, the proof of Proposition 2 follows from a rather direct perturbation argument. Notice that the critical case $\lambda_0 = 0$ is much more subtle, since more sensitive to perturbations, and left open here.

We now focus on the case $\lambda_0 < 0$, for which survival is expected when $|\varepsilon| \ll 1$. We thus inquire for nonnegative and nontrivial steady state $n = n^\varepsilon(x, y)$ solving

$$\partial_{xx}n + \partial_{yy}n + n\left(r\left(y - \varepsilon\theta(x)\right) - \int_{\mathbb{R}} n(x, y')\, dy'\right) = 0. \tag{7}$$

Notice that, in this paper, we reserve the notations $n = n^\varepsilon(x, y)$ to steady states and $u = u^\varepsilon(t, x, y)$ to time dependent solutions. Observe first that, when $\varepsilon = 0$, an appropriate renormalization of the ground state $\Gamma_0 = \Gamma_0(y)$ provides a positive solution: it is obvious that

$$n^0(y) := \eta\Gamma_0(y), \quad \eta := \frac{-\lambda_0}{\|\Gamma_0\|_{L^1}} > 0, \tag{8}$$

solves (7) when $\varepsilon = 0$. Our first main result is concerned with the construction of steady states when $|\varepsilon|$ is small enough.

Theorem 3 (Steady states)**.** *Assume $\lambda_0 < 0$. Let $\theta \in C_b(\mathbb{R})$. Let us fix $\beta > \frac{13}{4}$ and define the function space $Y = Y_\beta$ given by (29), equipped with the norm (31).*

Then there are $\varepsilon_0 > 0$ and $r_1 > 0$ such that, for any $|\varepsilon| < \varepsilon_0$,

$$\textit{there is a unique } n^\varepsilon \in Y \textit{ such that } \|n^\varepsilon - n^0\|_Y \le r_1 \textit{ and } n^\varepsilon \textit{ solves (7).} \tag{9}$$

Additionally, we have

$$n^\varepsilon = n^0 + \varepsilon n^1 + o(\varepsilon) \quad \textit{in } Y, \textit{ as } \varepsilon \to 0, \tag{10}$$

where

$$n^1 = n^1(x, y) := A(\rho_A * \theta)(x)yn^0(y), \tag{11}$$

with ρ_A the probability density given by

$$\rho_A(z) := \frac{1}{2}\sqrt{2A}e^{-\sqrt{2A}|z|}. \tag{12}$$

If we assume further that $\theta \in C_b^m(\mathbb{R})$ for some $m \geq 1$, then the same conclusions hold true with Y replaced by Y_m given by (53).

The proof relies on rigorous perturbation techniques, and involves rather intricate function spaces, such as $Y \subsetneq C_b^2(\mathbb{R}^2)$, $Y_m \subsetneq C_b^{2+m}(\mathbb{R}^2)$, which are precisely defined in (29), (53).

The role of β in the above result is the following. As explained previously, we look after a solution in the form of a Hilbert series, see (41). After finding the coordinates, we need to "reconstruct" the solution and prove it belongs to Y (regularity and decay estimates). This requires $\beta > \frac{13}{4}$ to ensure the uniform convergence of some series, see the end of subsection 4.2.

The positivity of the constructed steady state is not provided by our proof. Nevertheless, it can be proved *a posteriori* in some prototype situations for the perturbation $\theta = \theta(x)$, in particular when it is periodic or *localized*. To state this, we denote

$$C_{per}^L(\mathbb{R}) := \{f \in C(\mathbb{R}) : \forall x \in \mathbb{R}, f(x+L) = f(x)\}, \quad \text{for } L > 0,$$

and

$$C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : \lim_{|x|\to+\infty} f(x) = 0\}.$$

Theorem 4 (Positive steady states, the periodic case)**.** *Let the conditions of Theorem 3 hold and assume further that $\theta \in C_{per}^L(\mathbb{R})$ for some $L > 0$. Then, the steady states $n^\varepsilon = n^\varepsilon(x,y)$ constructed in Theorem 3 are L-periodic with respect to the x variable. Furthermore, up to reducing $\varepsilon_0 > 0$, there holds*

$$\forall a > 0, \exists C > 0, \forall |\varepsilon| < \varepsilon_0, \forall (x,y) \in \mathbb{R}^2, \quad 0 < n^\varepsilon(x,y) \leq Ce^{-a|y|}. \tag{13}$$

Theorem 5 (Positive steady states, the localized case)**.** *Let the conditions of Theorem 3 hold and assume further that $\theta \in C_0(\mathbb{R})$. Then, the steady states $n^\varepsilon = n^\varepsilon(x,y)$ constructed in Theorem 3 satisfy $n^\varepsilon - n_0 \in \tilde{Y}$, where the function space $\tilde{Y}$ is given by (56) and equipped with the norm (57). In particular,*

$$n^\varepsilon(x,y) \to n^0(y), \quad \text{as } |x| \to +\infty, \text{ uniformly w.r.t. } y \in \mathbb{R}. \tag{14}$$

Furthermore, up to reducing $\varepsilon_0 > 0$, there holds

$$\forall a > 0, \exists C > 0, \forall |\varepsilon| < \varepsilon_0, \forall (x,y) \in \mathbb{R}^2, \quad 0 < n^\varepsilon(x,y) \leq Ce^{-a|y|}. \tag{15}$$

The distortion of the positive steady state by the nonlinear (periodic or localized) perturbation $\theta = \theta(x)$ is encoded in (10)—(11) and will be discussed in details in subsections 6.1 an 6.2.

Next, still assuming $\lambda_0 < 0$, we enquire on the existence of fronts for equation (5). To deal with the $\varepsilon = 0$ case, let us recall the well-known fact concerning the Fisher-KPP traveling fronts: for any

$$c_0 \geq c_0^* := 2\sqrt{-\lambda_0} > 0,$$

there is a unique (up to translation) profile $U = U(z)$ solving

$$\begin{cases} U'' + c_0 U' - \lambda_0 U(1-U) = 0 \quad \text{on } \mathbb{R}, \\ U(-\infty) = 1, \\ U(+\infty) = 0. \end{cases} \tag{16}$$

which moreover satisfies $U' < 0$. Equipped with a Fisher-KPP front (c_0, U), a straightforward computation shows that, when $\varepsilon = 0$,

$$u^0(x - c_0 t, y) := U(x - c_0 t) n^0(y)$$

solves (5), where n^0 is the ground state given by (8). As explained above, this corresponds to a separation of the variables $z = x - c_0 t$ and y. In other words, the profile $n^0(y)$ invades the trivial state along the x axis at the spreading speed c_0.

Our second main result is concerned with the case $\theta \in C^L_{per}(\mathbb{R})$, for which we construct fronts when $|\varepsilon|$ is small enough. Because of the periodic term $\varepsilon\theta(x)$ in (5), we look for a pulsating front of the form $u^\varepsilon(x - c_\varepsilon t, x, y)$ with $u^\varepsilon = u^\varepsilon(z, x, y)$ satisfying

$$\begin{cases} u^\varepsilon(z, \cdot, y) \in C^L_{per}(\mathbb{R}) & \forall (z, y) \in \mathbb{R}^2, \\ u^\varepsilon(-\infty, x, y) = n^\varepsilon(x, y) & \text{uniformly w.r.t. } (x, y) \in \mathbb{R}^2, \\ u^\varepsilon(+\infty, x, y) = 0 & \text{uniformly w.r.t. } (x, y) \in \mathbb{R}^2, \end{cases} \tag{17}$$

where $n^\varepsilon = n^\varepsilon(x, y)$ is the (periodic in x) steady state provided by Theorem 4. That is, u^ε spreads at the perturbed speed c_ε and connects the steady state n^ε to the trivial one.

Theorem 6 (Pulsating fronts)**.** *Assume $\lambda_0 < 0$. Let us fix $c_0 \geq c_0^* := 2\sqrt{-\lambda_0}$ and consider $U = U(z)$ the unique (up to translation) profile solving (16). Let us fix $\beta > \frac{19}{4}$ and $\gamma > 3$. Assume $\theta \in C^{k,\delta}(\mathbb{R}) \cap C^L_{per}(\mathbb{R})$ with $L > 0$ and where $k \in \mathbb{N}$, $0 \leq \delta < 1$ satisfy $k + \delta > \gamma + \frac{1}{2}$. Let n^ε be the steady state solving (7) and obtained from Theorem 4.*

Then there is $\overline{\varepsilon}_0 > 0$ such that, for any $|\varepsilon| < \overline{\varepsilon}_0$, there are a speed $c_\varepsilon > 0$ and a profile $u^\varepsilon = u^\varepsilon(z, x, y) \in C^2_b(\mathbb{R}^3)$ such that

$$\begin{cases} u^\varepsilon \text{ satisfies (17)}, \\ (t, x, y) \mapsto u^\varepsilon(x - c_\varepsilon t, x, y) \text{ solves (5)}. \end{cases}$$

Additionally, we have

$$|c_\varepsilon - c_0| + \sup_{(z,x,y) \in \mathbb{R}^3} \left| (1 + y^2) e^{b|z|} \Big(u^\varepsilon(z, x, y) - U(z) n^\varepsilon(x, y) \Big) \right| \to 0, \ \textit{as } \varepsilon \to 0. \tag{18}$$

for some $b > 0$.

An inherent difficulty to the construction of pulsating fronts is that the underlying elliptic operator, see (64), is degenerate. This requires to consider a regularization, see (65), via a parameter $0 < \mu \ll 1$. For a fixed such μ, we use rigorous perturbation techniques (from the $\varepsilon = 0$ situation), that involve very intricate function spaces, which are precisely defined in Section 5. To deal with the phenotypic trait variable y we take advantage of a Hilbert basis of $L^2(\mathbb{R})$ made of eigenfunctions of an underlying Schrödinger operator, whereas to deal with the space variable x we use the Fourier series expansions. Last, thanks to a judicious choice of function spaces, we can let the regularization parameter $\mu \to 0$ and then catch the desired pulsating front solution for a *nontrivial* range of small $|\varepsilon|$. We refer to Remark 15 for more technical and precise details. Notice also that the uniqueness is lost through the $\mu \to 0$ limit.

Let us comment on the issue of the positivity of the constructed pulsating front which is not provided by our proof. In a related but different framework, a kinetic equation as a perturbation of a Fisher-KPP equation was considered in [20], see also [17]. Based on some stability results for the Cauchy problem, the authors of [20] recover *a posteriori* the positivity of super-critical traveling waves, while in [17] the positivity follows from the construction. However, such strategies heavily rely on the comparison principle which does not hold true for problem (1). One might be tempted to adapt the argument of subsection 4.5 which proves *a posteriori* the positivity of the constructed steady state, but this would require a precise control of the tail of the front as $z \to +\infty$, which is not reachable by our construction, nor by an adjustment of it. Nevertheless, we believe that a precise *a priori* argument, in the spirit of [29], may connect the "positivity issue" with some "minimal speed issue" denoted c_ε^*. Equipped with this, we conjecture that, up to reducing $\varepsilon_0 > 0$, one may prove *a posteriori* the positivity of the constructed pulsating front as soon as $c_0 > c_0^* = 2\sqrt{-\lambda_0}$. In other words, the positivity should not be lost, at least when we perturb from a *super-critical* traveling front. This is a very delicate issue, that would require lengthy arguments, and left here as an open question. However, this conjecture is supported by some numerical simulations in subsection 6.4.

Our analysis, see subsection 6.3, reveals that

$$c_\varepsilon = c_0 + o(\varepsilon), \qquad \text{as } \varepsilon \to 0. \tag{19}$$

In other words, the perturbation of the speed of the front by the periodic nonlinearity $\theta = \theta(x)$ vanishes *at the first order with respect to* ε, which is a relevant biological information. This could be guessed from the observation that replacing (y, ε) with $(-y, -\varepsilon)$ leaves the model unchanged. We also conjecture $c_\varepsilon < c_0$ when $0 \neq |\varepsilon| \ll 1$, which is supported by some numerical explorations in subsection 6.4. On the other hand, the distortion of the profile of the front is less predictable, but our mathematical analysis provides some clues. We refer to Example 32 in subsection 6.3.

The organization of the paper is as follows. In Section 3, we prove the extinction result, namely Proposition 2, and present some useful tools for the following, in particular some spectral properties. The steady states are constructed in Section 4 through the proofs of Theorem 3, Theorem 4 and Theorem 5. In Section 5, we construct pulsating fronts by proving Theorem 6. Last, in Section 6, we present some biological insights of our results, together with some numerical explorations.

3 Preliminaries

3.1 Extinction result

We here consider the case $\lambda_0 > 0$ for which we prove extinction, as stated in Proposition 2.

Proof of Proposition 2. For $M \geq 0$ given by (6), we consider

$$\phi(t, y) := M e^{-\mu_0 t} e^{-\frac{1}{2} a y^2}$$

which satisfies

$$\partial_t \phi - \partial_{xx}\phi - \partial_{yy}\phi - r(y - \varepsilon\theta(x))\phi = \left[(A^2 - a^2)y^2 - 2\varepsilon A^2 \theta(x) y + \varepsilon^2 A^2 \theta^2(x) + a - 1 - \mu_0\right] \phi.$$

The discriminant of the quadratic polynomial in y is $4A^2a^2\theta^2(x)\varepsilon^2 - 4(A^2 - a^2)(a - 1 - \mu_0)$, which is uniformly (with respect to $x \in \mathbb{R}$) negative for $|\varepsilon| \leq \varepsilon_0$ for $\varepsilon_0 > 0$ sufficiently small (recall that θ is bounded). As a result

$$\partial_t \phi - \partial_{xx}\phi - \partial_{yy}\phi - r(y - \varepsilon\theta(x))\phi \geq 0 \geq \partial_t u - \partial_{xx} u - \partial_{yy} u - r(y - \varepsilon\theta(x))u.$$

Since we know from (6) that $\phi(0, y) \geq u_0(x, y)$, we deduce from the comparison principle that $u(t, x, y) \leq \phi(t, y)$, which concludes the proof. □

3.2 Implicit Function Theorem

We recall the *Implicit Function Theorem*, see [50, Theorem 4.B] for instance.

Theorem 7 (Implicit Function Theorem)**.** *Let X, Y, Z be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with their respective norms $||.||_X$, $||.||_Y$ and $||.||_Z$. Let U be a open neighborhood of $(0, 0)$ in $X \times Y$. Let $F\colon U \to Z$ be a map. Suppose that*

(i) $F(0, 0) = 0$, and F is continuous at $(0, 0)$,

(ii) $D_y F$ exists as a partial Fréchet derivative on U, and $D_y F$ is continuous at $(0, 0)$,

(iii) $D_y F(0, 0)\colon Y \to Z$ is bijective.

Then the following are true:

- *There are $r_0 > 0$ and $r_1 > 0$ such that, for every $x \in X$ satisfying $||x||_X < r_0$, there is a unique $y(x) \in Y$ for which $||y(x)||_Y \leq r_1$ and $F(x, y(x)) = 0$.*
- *If F is C^k on U with $0 \leq k \leq \infty$, then $x \mapsto y(x)$ is also C^k on a neighborhood of 0.*

3.3 Linear material

In this subsection we fix $A > 0$ and consider the operator $\mathcal{H}w := -w'' - (1 - A^2y^2)w$, which corresponds to the harmonic oscillator. The following is well-known.

Proposition 8 (Eigenelements of the harmonic oscillator)**.** *The operator $\mathcal{H}w := -w'' - (1 - A^2y^2)w$ admits a family of eigenelements $(\lambda_i, \Gamma_i)_{i\in\mathbb{N}}$, where*

$$\lambda_i = -1 + (2i+1)A, \tag{20}$$

and $\Gamma_i(y) = C_i H_i(\sqrt{A}y)e^{-\frac{1}{2}Ay^2}$. Here $(H_i)_{i\in\mathbb{N}}$ denotes the family of Hermite polynomials, that is the unique family of real polynomials satisfying

$$\int_{\mathbb{R}} H_i(x)H_j(x)e^{-x^2}dx = 2^i i!\sqrt{\pi}\delta_{ij}, \qquad \deg H_i = i,$$

and

$$C_i = \left(\frac{A}{\pi}\right)^{1/4}\sqrt{\frac{1}{2^i i!}} \tag{21}$$

a normalization constant so that $||\Gamma_i||_{L^2} = 1$.

Additionally, the family $(\Gamma_i)_{i\in\mathbb{N}}$ forms a Hilbert basis of $L^2(\mathbb{R})$.

We now present some relations between the eigenfunctions, which will prove useful in our proofs.

Lemma 9 (Some linear relations)**.** *For any integer i, there holds*

$$y\Gamma_i(y) = p_i^+\Gamma_{i+1}(y) + p_i^-\Gamma_{i-1}(y), \qquad p_i^+ := \sqrt{\frac{i+1}{2A}}, \qquad p_i^- := \sqrt{\frac{i}{2A}}. \tag{22}$$

and

$$\Gamma_i'(y) = q_i^+\Gamma_{i+1}(y) + q_i^-\Gamma_{i-1}(y), \qquad q_i^+ = -\sqrt{\frac{(i+1)A}{2}}, \qquad q_i^- = \sqrt{\frac{iA}{2}}. \tag{23}$$

with the conventions $p_0^-\Gamma_{-1}(y) \equiv q_0^-\Gamma_{-1}(y) \equiv 0$.

Proof. The Hermite polynomials are known to satisfy the recursion relation

$$2xH_i(x) = H_{i+1}(x) + 2iH_{i-1}(x).$$

Multiplying this by C_i and setting $x = \sqrt{A}y$, we get $2\sqrt{A}y\Gamma_i(y) = \frac{C_i}{C_{i+1}}\Gamma_{i+1}(y) + \frac{2iC_i}{C_{i-1}}\Gamma_{i-1}(y)$ which, combined with (21), proves (22).

The Hermite polynomials are known to satisfy the relations

$$H_i'(x) = 2iH_{i-1}(x), \qquad 2xH_i(x) = H_{i+1}(x) + 2iH_{i-1}(x).$$

Differentiating the expression $\Gamma_i(y) = C_iH_i(\sqrt{A}y)e^{-\frac{1}{2}Ay^2}$ and using the above relations, we reach

$$\Gamma_i'(y) = C_i\left(i\sqrt{A}H_{i-1}(\sqrt{A}y) - \frac{1}{2}\sqrt{A}H_{i+1}(\sqrt{A}y)\right)e^{-\frac{1}{2}Ay^2} = \frac{i\sqrt{A}C_i}{C_{i-1}}\Gamma_{i-1}(y) - \frac{\sqrt{A}C_i}{2C_{i+1}}\Gamma_{i+1}(y),$$

which, combined with (21), proves (23). □

We pursue with some L^∞ and L^1 estimates on eigenfunctions, possibly with some polynomial weight.

Lemma 10 (L^∞ and L^1 estimates)**.** *There is $C = C(A) > 0$ such that, for all $i \in \mathbb{N}$,*

$$||\Gamma_i||_{L^1} \leq Ci^{1/4}, \tag{24}$$

$$||\Gamma_i||_{L^\infty} \leq Ci^{1/4}, \tag{25}$$

together with

$$||\Gamma_i'||_{L^\infty} \leq Ci^{3/4}, \quad ||\Gamma_i''||_{L^\infty} \leq Ci^{5/4}, \tag{26}$$

and

$$||y^2\Gamma_i||_{L^\infty} \leq Ci^{5/4}, \quad ||y^4\Gamma_i||_{L^\infty} \leq Ci^{9/4}. \tag{27}$$

Proof. The non so standard L^1 estimate (24) can be found in [5, Proposition 2.4], whereas the L^∞ estimate (25) can be found in [5, Proposition 2.6]. Next, estimate (26) easily follows from the combination of (23) and (25), whereas estimate (27) easily follows from (22) and (25). Details are omitted. □

Throughout this paper, we denote m_i the "mass" of the i-th eigenfunction, namely

$$m_i := \int_{\mathbb{R}} \Gamma_i(y)dy. \tag{28}$$

4 Construction of steady states

In this section, we prove Theorem 3 on steady states, Theorem 4 on the periodic case, and Theorem 5 on the localized case.

We look after a steady state solution to (7) in the perturbative form $n^\varepsilon(x,y) = n^0(y) + h^\varepsilon(x,y)$, where $n^0 = n^0(y)$, given by (8), is a steady state when $\varepsilon = 0$. From straightforward computations, we are left to find h^ε satisfying $F(\varepsilon, h^\varepsilon) = 0$, where

$$\begin{aligned} F(\varepsilon, h) := h_{xx} + h_{yy} + n^0 &\left(2A^2\varepsilon\theta(x)y - A^2\varepsilon^2\theta^2(x) - \int_{\mathbb{R}} h(x,y')dy'\right) \\ &+ h\left(1 - A^2(y-\varepsilon\theta(x))^2 + \lambda_0 - \int_{\mathbb{R}} h(x,y')dy'\right). \end{aligned}$$

We thus aim at applying the Implicit Function Theorem, namely Theorem 7, to $F\colon \mathbb{R}\times Y \to Z$ where the function spaces Y, Z are to be appropriately chosen.

4.1 Function spaces

Let us fix $\beta > \frac{13}{4}$. Recall that $\Gamma_i = \Gamma_i(y)$ are the eigenfunctions defined in subsection 3.3. We set

$$Y := \left\{ h \in C^2(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C>0,\ \forall|\alpha|\le 2, \quad |D^\alpha h(x,y)| \le \frac{C}{(1+y^2)^2} \quad \text{on } \mathbb{R}^2, \\ \exists K>0,\ \forall k\le 2,\ \forall i\in\mathbb{N}, \quad \sup\limits_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} D_x^k h(x,y)\Gamma_i(y)dy\right| \le \frac{K}{(1+i)^{\beta+1-k/2}} \end{array} \right\}, \tag{29}$$

and

$$Z := \left\{ f \in C(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C>0, \quad |f(x,y)| \le \frac{C}{1+y^2} \quad \text{on } \mathbb{R}^2, \\ \exists K>0, \forall i\in\mathbb{N}, \quad \sup\limits_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} f(x,y)\Gamma_i(y)dy\right| \le \frac{K}{(1+i)^\beta} \end{array} \right\}, \tag{30}$$

equipped with the norms

$$||h||_Y := \sum_{|\alpha|\le 2} \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)^2 D^\alpha h(x,y)\right| + \sum_{k=0}^{2} ||D_x^k h||_{\beta+1-k/2}, \tag{31}$$

and

$$||f||_Z := \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2) f(x,y)\right| + ||f||_\beta, \tag{32}$$

where, for $m \in \mathbb{R}$, we define

$$||w||_m := \sup_{i\in\mathbb{N}} \left[(1+i)^m \sup_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} w(x,y)\Gamma_i(y)dy\right|\right]. \tag{33}$$

Remark 11 (Choice of the function spaces)**.** *Let us comment on the spaces Y, Z and the two controls appearing in their definition. The crux of the proof is to show that $L := D_hF(0,0)$, given by (40), is bijective from Y to Z: for every fixed $f \in Z$, there is a unique $h \in Y$ such that $Lh = f$. First, thanks to the controls on the y-tails, i.e. the first constraint in the definition of Y and Z, $h(x,\cdot), f(x,\cdot) \in L^2(\mathbb{R})$ for all $x \in \mathbb{R}$. This allows to decompose h and f along the eigenfunction basis $(\Gamma_i)_{i\in\mathbb{N}}$, leading to*

(41). From there we obtain an expression of $h_i = h_i(x)$ given by (46)—(47). Next, the control on $f_i(x) = \int_{\mathbb{R}} f(x,y)\Gamma_i(y)dy$, i.e. the second constraint in the definition of Z, allows to prove the bounds (49)—(51) for h_i. This in turn allows to prove the control on the y-tails for h. This is done by using (25)—(27) and by taking $\beta > \frac{13}{4}$.

In what follows, it is useful to keep in mind the straightforward estimates

$$\forall h \in Y, \forall |\alpha| \leq 2, \forall (x,y) \in \mathbb{R}^2, \quad \left| D^k h(x,y) \right| \leq \frac{||h||_Y}{(1+y^2)^2}, \tag{34}$$

$$\forall f \in Z, \forall (x,y) \in \mathbb{R}^2, \quad |f(x,y)| \leq \frac{||f||_Z}{1+y^2}, \tag{35}$$

and

$$\forall h \in Y, \forall x \in \mathbb{R}, \quad \left| \int_{\mathbb{R}} h(x,y)dy \right| \leq ||h||_Y \int_{\mathbb{R}} (1+y^2)^{-2} dy = \frac{\pi}{2} ||h||_Y. \tag{36}$$

Lemma 12 (Y and Z are Banach)**.** *The spaces Y given by (29), and Z given by (30), are Banach spaces when equipped with their respective norm $||.||_Y$ given by (31), and $||.||_Z$ given by (32).*

Proof. For the sake of completeness, let us give a short proof that Y is Banach, the proof for Z being similar. Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. Since the injection $Y \hookrightarrow C_b^2(\mathbb{R}^2)$ is continuous and $C_b^2(\mathbb{R}^2)$ is Banach, there is $h \in C_b^2(\mathbb{R}^2)$ such that $h_n \to h$ in the norm $||.||_{C_b^2(\mathbb{R}^2)}$.

Let us prove that $h \in Y$. Set

$$C_n := \sum_{|\alpha|\leq 2} \sup_{(x,y)\in\mathbb{R}^2} \left| (1+y^2)^2 D^\alpha h_n(x,y) \right|.$$

Since (h_n) is Cauchy, the sequence $(C_n)_{n\in\mathbb{N}}$ is bounded by some $C \geq 0$. Then, for all $|\alpha| \leq 2$ and $(x,y) \in \mathbb{R}^2$, there holds $|D^\alpha h_n(x,y)| \leq \frac{C}{(1+y^2)^2}$, and $n \to +\infty$ yields

$$|D^\alpha h(x,y)| \leq \frac{C}{(1+y^2)^2}.$$

Similarly, the sequence

$$K_n := \sum_{k\leq 2} ||D_x^k h_n||_{\beta+1-k/2}$$

is bounded by some $K \geq 0$. Then, for all $0 \leq k \leq 2$, there holds

$$\forall i \in \mathbb{N}, \forall x \in \mathbb{R}, \quad \left| \int_{\mathbb{R}} D_x^k h_n(x,y)\Gamma_i(y)dy \right| \leq \frac{K}{(1+i)^{\beta+1-k/2}}.$$

Given that $|D_x^k h_n(x,y)| \leq \frac{C}{(1+y^2)^2}$, the dominated convergence theorem allows to let $n \to +\infty$ and obtain that the above estimate also holds for h. We conclude that $h \in Y$.

Now, very classical arguments (that we omit) yield that $h_n \to h$ in Y. □

We conclude this subsection with a preliminary result, which in particular states that each "h-term" appearing in $F(\varepsilon, h)$ has its Z-norm controlled by the Y-norm of h. For better readability we denote yh and y^2h the functions $(x,y) \mapsto yh(x,y)$ and $(x,y) \mapsto y^2h(x,y)$ respectively.

Lemma 13 (Controlling in Z the terms of $F(\varepsilon,h)$)**.** *There is $C = C(A) > 0$ such that, for all $h \in Y$,*

$$\max\left(||h||_Z, ||yh||_Z, ||y^2h||_Z, ||h_{xx}||_Z, ||h_{yy}||_Z \right) \leq C||h||_Y. \tag{37}$$

Also, if $f = f(x,y) \in Z$ and $b = b(x) \in C_b(\mathbb{R})$, then

$$||bf||_Z \leq ||b||_{L^\infty} ||f||_Z. \tag{38}$$

Proof. The proof of assertion (38) is obvious. As for (37), the estimates for h and h_{xx} follow directly from the definitions of Y, Z and their respective norms. In the sequel, C denotes a positive constant that may change from line to line, but that always depends only on A.

Let us prove $||h_{yy}||_Z \leq C||h||_Y$. Since $\left|(1+y^2)h_{yy}(x,y)\right| \leq \left|(1+y^2)^2 h_{yy}(x,y)\right| \leq ||h||_Y$, it remains to consider the second term appearing in the right hand side of (32), that is $||h_{yy}||_\beta$. Integrating by parts and using (23) we have

$$\begin{aligned}\left|\int_{\mathbb{R}} h_{yy}(x,y)\Gamma_i(y)dy\right| &= \left|\int_{\mathbb{R}} h(x,y)\Gamma_i''(y)dy\right| \\ &\leq q_i^+ q_{i+1}^+ \left|\int_{\mathbb{R}} h\Gamma_{i+2}dy\right| + \left(q_i^+ q_{i+1}^- + q_i^- q_{i-1}^+\right)\left|\int_{\mathbb{R}} h\Gamma_i dy\right| + q_i^- q_{i-1}^- \left|\int_{\mathbb{R}} h\Gamma_{i-2}dy\right| \\ &\leq C\frac{\sqrt{(1+i)(2+i)}}{(1+i)^{\beta+1}}||h||_Y,\end{aligned}$$

from the expressions of $q_i^\pm$ and the fact that $h \in Y$ so that $||h||_{\beta+1} \leq ||h||_Y$. As a result $||h_{yy}||_\beta \leq C||h||_Y$ and we are done.

As for the cases of yh and y^2h, it suffices to use (22) instead of (23) and very similar arguments. □

4.2 Checking assumptions of Theorem 7

Equipped with the function spaces Y and Z, we thus consider

$$\begin{aligned}F(\varepsilon,h) := h_{xx} + h_{yy} + n^0\left(2A^2\varepsilon\theta(x)y - A^2\varepsilon^2\theta^2(x) - \int_{\mathbb{R}} h(x,y')dy'\right) \\ + h\left(1 - A^2(y-\varepsilon\theta(x))^2 + \lambda_0 - \int_{\mathbb{R}} h(x,y')dy'\right).\end{aligned} \tag{39}$$

Clearly $F(0,0)=0$. We prove below that the assumptions of Theorem 7 hold true.

Checking assumptions (i) *and* (ii) *of Theorem 7.* We first check that F is well defined. Recalling that $n^0(y) = \eta\Gamma_0(y)$ and since $(\Gamma_i)_{i\in\mathbb{N}}$ is orthonormal in $L^2(\mathbb{R})$, it is clear that the conditions in (30) are satisfied, so that $n^0 \in Z$. Similarly and in view of (22), $yn^0 \in Z$. Next, for fixed $\varepsilon \in \mathbb{R}$ and $h \in Y$, the function $b(x) := -A^2\varepsilon^2\theta^2(x) - \int_{\mathbb{R}} h(x,y')dy'$ is continuous and bounded thanks to (36), and therefore $bn^0 \in Z$ from Lemma 13. In the same way, setting $\tilde{b}(x) := 2A^2\varepsilon\theta(x)$, we obtain $\tilde{b}yn^0 \in Z$. Finally, the other terms in $F(\varepsilon,h)$ also belong to Z, again by virtue of Lemma 13.

We now compute $D_hF(0,0)$ the Fréchet derivative of F along the second variable at point $(0,0)$. We have $F(0,h) = Lh + R(h)$, where

$$Lh := h_{xx} + h_{yy} + h\left(1 - A^2y^2 + \lambda_0\right) - n^0\int_{\mathbb{R}} h(x,y')dy', \tag{40}$$

and $R(h) = -h\int_{\mathbb{R}} h(x,y')dy'$. From Lemma 13 and (36), the remainder $R(h)$ satisfies

$$||R(h)||_Z \leq ||h||_Z \left\|\int_{\mathbb{R}} h(\cdot,y')dy'\right\|_{L^\infty} \leq C||h||_Y^2.$$

On the other hand, $L\colon Y \to Z$ is a linear continuous operator, which is readily seen by using Lemma 13 and (36). Since $F(0,0)=0$, we then have $D_hF(0,0) = L$.

Using similar arguments, one shows that D_hF is well-defined on a neighborhood of $(0,0)$, as well as the continuity of F and D_hF at $(0,0)$. □

Now, the main part is to prove the bijectivity of $D_hF(0,0) = L : Y \to Z$.

Checking assumption (iii) *of Theorem 7.* We proceed by analysis and synthesis. Let $f \in Z$ be given, and assume there exists $h \in Y$ such that $Lh = f$. Thanks to (34) and (35), $f(x,\cdot)$ and $h(x,\cdot)$ are in

$L^2(\mathbb{R})$ for any $x \in \mathbb{R}$. Since the family of eigenfunctions $(\Gamma_i)_{i\in\mathbb{N}}$ of Proposition 8 forms a Hilbert basis of $L^2(\mathbb{R})$, we can write

$$h(x,y) = \sum_{i=0}^{+\infty} h_i(x)\Gamma_i(y), \quad f(x,y) = \sum_{i=0}^{+\infty} f_i(x)\Gamma_i(y), \tag{41}$$

where, for any $i \in \mathbb{N}$,

$$h_i(x) := \int_{\mathbb{R}} h(x,y)\Gamma_i(y)dy, \quad f_i(x) := \int_{\mathbb{R}} f(x,y)\Gamma_i(y)dy.$$

Notice that, for any $x \in \mathbb{R}$, the equalities in (41) correspond, *a priori*, to a convergence of the series in the Hilbert space $L^2(\mathbb{R})$ norm. However, since $h \in Y$ and $f \in Z$, there holds

$$||h_i||_{L^\infty} \leq \frac{||h||_Y}{(1+i)^{\beta+1}}, \tag{42}$$

and

$$||f_i||_{L^\infty} \leq \frac{||f||_Z}{(1+i)^{\beta}}. \tag{43}$$

Consequently, since $\beta > \frac{13}{4} > \frac{5}{4}$ and (25) holds, the convergences in (41) are also valid pointwise in $\mathbb{R}^2$. Similarly, thanks to (24), the equality

$$\int_{\mathbb{R}} h(x,y)dy = \sum_{i=0}^{+\infty} h_i(x) \int_{\mathbb{R}} \Gamma_i(y)dy$$

holds pointwise in $\mathbb{R}$. Also, thanks to (34) and (35), we obtain that $h_i \in C_b^2(\mathbb{R})$ and $f_i \in C_b(\mathbb{R})$, with

$$h_i'(x) = \int_{\mathbb{R}} h_x(x,y)\Gamma_i(y)dy, \quad h_i''(x) = \int_{\mathbb{R}} h_{xx}(x,y)\Gamma_i(y)dy.$$

Now, we project equality $f = Lh$ on each Γ_i so that, for all $x \in \mathbb{R}$,

$$\begin{aligned}
f_i(x) &= \int_{\mathbb{R}} h_{xx}(x,y)\Gamma_i(y)dy + \int_{\mathbb{R}} h_{yy}(x,y)\Gamma_i(y)dy + \int_{\mathbb{R}} (1 - A^2y^2 + \lambda_0)h(x,y)\Gamma_i(y)dy \\
&\quad - \left(\int_{\mathbb{R}} h(x,y')dy'\right) \int_{\mathbb{R}} n^0(y)\Gamma_i(y)dy \\
&= h_i''(x) + \int_{\mathbb{R}} h(x,y)\left[\Gamma_i''(y) + (1 - A^2y^2 + \lambda_0)\Gamma_i(y)\right] dy - \eta\delta_{i0} \int_{\mathbb{R}} h(x,y')dy' \\
&= h_i''(x) - (\lambda_i - \lambda_0)h_i(x) - \eta\delta_{i0} \sum_{i=0}^{+\infty} h_i(x) \int_{\mathbb{R}} \Gamma_i(y)dy,
\end{aligned}$$

where we have integrated by parts and used (8). Therefore, $Lh = f$ is reduced to an infinite system of linear ordinary differential equations for the h_i's, namely

$$h_i'' - (\lambda_i - \lambda_0)h_i = f_i(x), \quad (i \geq 1), \tag{44}$$

and

$$h_0'' + \lambda_0 h_0 = f_0(x) + \eta \sum_{i=1}^{+\infty} m_i h_i(x), \tag{45}$$

where we recall the notation (28) for the mass m_i. Notice that, combining (42) with (24), the series appearing in the right-hand side of (45) converges to a function in $C_b(\mathbb{R})$.

We first deal with the case $i \geq 1$, that is (44). Since $\lambda_i - \lambda_0 > 0$ and $f_i \in C_b(\mathbb{R})$, there is a unique solution h_i to (44) which remains in $C_b^2(\mathbb{R})$, and it is explicitly given by

$$h_i(x) = -\rho_i * f_i(x) \;\text{ where }\; \rho_i(z) := \frac{1}{2\sqrt{\lambda_i - \lambda_0}} e^{-\sqrt{\lambda_i - \lambda_0}|z|}, \quad (i \geq 1). \tag{46}$$

The functions h_i $(i \geq 1)$ now determined, we can deal with the $i = 0$ case. Since $\lambda_0 < 0$, there is a unique solution h_0 to (45) which remains in $C_b^2(\mathbb{R})$, and it is explicitly given by

$$h_0(x) = -\rho_0 * \left(f_0 + \eta \sum_{i=1}^{+\infty} m_i h_i\right)(x) \quad \text{where} \quad \rho_0(z) := \frac{1}{2\sqrt{-\lambda_0}} e^{-\sqrt{-\lambda_0}|z|}. \tag{47}$$

Conversely, we need to prove that, for $h_i = h_i(x)$ provided by (46) and then (47), the function

$$h(x,y) := \sum_{i=0}^{+\infty} h_i(x)\Gamma_i(y), \tag{48}$$

does belong to Y and that $Lh = f$.

Let us first prove that $h \in C^2(\mathbb{R}^2)$. In the sequel, C denotes a positive constant that may change from line to line, but that always depends only on A and $||f||_Z$. From (43) and (20) we deduce that, for all $i \geq 1$,

$$||h_i||_{L^\infty} \leq ||\rho_i||_{L^1}||f_i||_{L^\infty} \leq \frac{1}{\lambda_i - \lambda_0} \times \frac{||f||_Z}{(1+i)^\beta} \leq \frac{C}{(1+i)^{\beta+1}}, \tag{49}$$

$$||h_i'||_{L^\infty} \leq ||\rho_i'||_{L^1}||f_i||_{L^\infty} \leq \frac{1}{\sqrt{\lambda_i - \lambda_0}} \times \frac{||f||_Z}{(1+i)^\beta} \leq \frac{C}{(1+i)^{\beta+1/2}}, \tag{50}$$

and thus, from equation (44),

$$||h_i''||_{L^\infty} \leq ||f_i||_{L^\infty} + (\lambda_i - \lambda_0)||h_i||_{L^\infty} \leq \frac{C}{(1+i)^\beta}. \tag{51}$$

Therefore, with (25), the series in (48) is normally convergent, and the equality is valid pointwise. Now, since $\beta > \frac{13}{4} > \frac{5}{4}$, combining (49)—(51) and (25)—(26), we obtain that $h \in C^2(\mathbb{R}^2)$, with the pointwise expressions

$$D_x^p D_y^q h(x,y) = \sum_{i=0}^{+\infty} \frac{d^p h_i}{dx^p}(x) \frac{d^q \Gamma_i}{dy^q}(y), \quad (p+q \leq 2). \tag{52}$$

Also, recalling definition (33), we infer from (49)—(51) that

$$\sum_{k=0}^{2} ||D_x^k h||_{\beta+1-k/2} < +\infty.$$

In view of (31), we now need to prove that $(x,y) \mapsto (1+y^2)^2 D^\alpha h(x,y)$ is bounded for any multi-index $|\alpha| \leq 2$. Using (49) and (26), we find that, for all $(x,y) \in \mathbb{R}^2$,

$$\left|(1+y^2)^2 h(x,y)\right| \leq \sum_{i=0}^{+\infty} |h_i(x)| \times \left|(1+2y^2+y^4)\Gamma_i(y)\right| \leq C \sum_{i=0}^{+\infty} \frac{i^{9/4}}{(1+i)^{\beta+1}} < +\infty,$$

since $\beta > \frac{13}{4} > \frac{9}{4}$. Analogously, combining (49)—(51) with (26), we can deal with $D^\alpha h$ for any other multi-index $|\alpha| \leq 2$. For instance, notice the so-called "worst case":

$$\left|(1+y^2)^2 h_{xx}(x,y)\right| \leq C \sum_{i=0}^{+\infty} \frac{i^{9/4}}{(1+i)^\beta} < +\infty,$$

since $\beta > 13/4$.

Eventually, we proved that $||h||_Y < +\infty$, therefore $h \in Y$ and the proof of $Lh = f$ is clear. $\square$

4.3 Completion of the proof of Theorem 3

Proof of Theorem 3. From the above two subsections, we can apply Theorem 7 to the function F around the point $(0,0)$. Hence there are $\varepsilon_0 > 0$ and $r_1 > 0$ such that, for any $|\varepsilon| < \varepsilon_0$, the following holds: there is a unique $h^\varepsilon \in Y$ for which $||h^\varepsilon||_Y \le r_1$ and $F(\varepsilon, h^\varepsilon) = 0$. Recalling $n^\varepsilon(x,y) = n^0(y) + h^\varepsilon(x,y)$, this transfers into (9).

Let us now prove (10). Since F is of the class C^1 (the case of the variable h was treated in subsection 4.2 and the case of the ε variable is clear) we deduce from Theorem 7, $F(\varepsilon, h^\varepsilon) = 0$ and the chain rule that

$$D_\varepsilon F(\varepsilon, h^\varepsilon) + D_h F(\varepsilon, h^\varepsilon)\left(\frac{dh^\varepsilon}{d\varepsilon}\right) = 0,$$

which we evaluate at $\varepsilon = 0$ to get

$$D_\varepsilon F(0,0) + L\left(\frac{dh^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0}\right) = 0.$$

From the expression of $F = F(\varepsilon, h)$ we easily compute $D_\varepsilon F(0,0) = 2A^2\theta(x)yn^0(y)$, so that, since $n^0(y) = \eta\Gamma_0(y)$,

$$\frac{dh^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = -2A^2L^{-1}\left(\theta(x)yn^0(y)\right) = -2A^2\eta L^{-1}\left(\theta(x)y\Gamma_0(y)\right).$$

From (22) we know $y\Gamma_0(y) = \frac{1}{\sqrt{2A}}\Gamma_1(y)$ so that

$$\frac{dh^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = -\sqrt{2}A^{3/2}\eta L^{-1}\left(\theta(x)\Gamma_1(y)\right).$$

Now, we compute $L^{-1}\left(\theta(x)\Gamma_1(y)\right)$ via (46) and (47) and reach (recall that $m_1 = \int_{\mathbb{R}}\Gamma_1(y)dy = 0$)

$$\begin{aligned}\frac{dh^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} &= -\sqrt{2}A^{3/2}\eta\left[\eta m_1\left(\rho_0 * (\rho_1 * \theta)\right)(x)\Gamma_0(y) - (\rho_1 * \theta)(x)\Gamma_1(y)\right] \\ &= \sqrt{2}A^{3/2}\left[y\sqrt{2A}(\rho_1 * \theta)(x)\right]n_0(y) \\ &= 2A^2(\rho_1 * \theta)(x)\, yn_0(y),\end{aligned}$$

which can be recast (10).

It remains to consider the case when we assume further that $\theta \in C_b^m(\mathbb{R})$ for some $m \ge 1$ which, in particular, improves the regularity of the solution $n^\varepsilon = n^\varepsilon(x,y)$. In this case one can actually redo the proofs above by replacing the spaces Y, Z in (29) and (30) with Y_m, Z_m given by

$$Y_m := \left\{ h \in C^{m+2}(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C > 0, \forall |\alpha| \le m+2, \quad |D^\alpha h(x,y)| \le \frac{C}{(1+y^2)^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0, \forall k \le m+2,\, \forall i \in \mathbb{N}, \\ \sup_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} D_x^k h(x,y)\Gamma_i(y)dy\right| \le \frac{K}{(1+i)^{\beta+(m+2-k)/2}} \end{array} \right\}, \tag{53}$$

and

$$Z_m := \left\{ f \in C^{m}(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C > 0, \forall |\alpha| \le m, \quad |D^\alpha f(x,y)| \le \frac{C}{1+y^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0, \forall k \le m,\, \forall i \in \mathbb{N}, \\ \sup_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} D_x^k f(x,y)\Gamma_i(y)dy\right| \le \frac{K}{(1+i)^{\beta+(m-k)/2}} \end{array} \right\},$$

equipped with their respective norms

$$||h||_{Y_m} := \sum_{|\alpha|\le m+2} \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)^2 D^\alpha h(x,y)\right| + \sum_{k=0}^{m+2} ||D_x^k h||_{\beta+(m+2-k)/2},$$

and

$$||f||_{Z_m} := \sum_{|\alpha|\le m} \sup_{x,y\in\mathbb{R}} \left|(1+y^2) D^\alpha f(x,y)\right| + \sum_{k=0}^{m} ||D_x^k f||_{\beta+(m-k)/2},$$

where we recall definition (33). Details are omitted. □

4.4 Additional properties in the periodic and localized cases

In this subsection, we start the proof of Theorem 4 and Theorem 5, estimates (13) and (15) being postponed to the next subsection.

Proof of the periodicity of the steady states in Theorem 4. In addition to the conditions of Theorem 3, let us assume $\theta \in C^L_{per}(\mathbb{R})$ for some $L > 0$. Let us recall that, from subsection 4.3, for any $|\varepsilon| < \varepsilon_0$, there is a unique $h^\varepsilon \in Y$ for which $||h^\varepsilon||_Y \le r_1$ and $F(\varepsilon, h^\varepsilon) = 0$. Defining

$$\tilde{h}^\varepsilon(x,y) := h^\varepsilon(x+L,y),$$

one readily checks that $F(\varepsilon,\tilde{h}^\varepsilon) = 0$ and $||\tilde{h}^\varepsilon||_Y \le r_1$. Therefore $h^\varepsilon \equiv \tilde{h}^\varepsilon$, that is h^ε is L-periodic in x, and so is $n^\varepsilon(x,y) = n^0(y) + h^\varepsilon(x,y)$. □

Proof of $n^\varepsilon - n_0 \in \tilde{Y}$, where $\tilde{Y}$ is defined by (56), in Theorem 5. In addition to the conditions of Theorem 3, let us assume $\theta \in C_0(\mathbb{R})$. Our proof relies on the following technical lemma, whose proof is postponed.

Lemma 14 (Function G). *Let $\theta \in C_0(\mathbb{R})$. Then there is a piecewise constant function $G > 0$ such that*

$$\begin{cases} G(x) \ge \max\left(|\theta(x)|, \theta^2(x)\right), & \forall x \in \mathbb{R}, \\ G \text{ is even on } \mathbb{R}, \text{ nonincreasing on } [0,+\infty), \\ \lim_{x\to+\infty} G(x) = 0, \end{cases} \tag{54}$$

together with the following property: there is $\sigma > 0$ such that, for all $i \in \mathbb{N}$,

$$(\rho_i * G)(x) \le \frac{\sigma}{1+i} G(x), \quad \forall x \in \mathbb{R}, \tag{55}$$

where the ρ_i's are given by (46) and (47).

Then, equipped with such a function G, we can redo the proof of subsections 4.1 to 4.3 by replacing the spaces Y, Z in (29) and (30) with

$$\tilde{Y} := \left\{ h \in C^2(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C > 0,\ \forall |\alpha| \le 2, \quad |D^\alpha h(x,y)| \le \frac{CG(x)}{(1+y^2)^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0,\ \forall k \le 2,\ \forall i \in \mathbb{N}, \quad \left|\int_{\mathbb{R}} D_x^k h(x,y)\Gamma_i(y)dy\right| \le \frac{KG(x)}{(1+i)^{\beta+1-k/2}} \quad \text{on } \mathbb{R} \end{array} \right\}, \tag{56}$$

and

$$\tilde{Z} := \left\{ f \in C(\mathbb{R}^2) \;\middle|\; \begin{array}{c} \exists C > 0, \quad |f(x,y)| \le \frac{CG(x)}{1+y^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0, \forall i \in \mathbb{N}, \quad \left|\int_{\mathbb{R}} f(x,y)\Gamma_i(y)dy\right| \le \frac{KG(x)}{(1+i)^\beta} \quad \text{on } \mathbb{R} \end{array} \right\},$$

equipped with the norms

$$||h||_{\tilde{Y}} := \sum_{|\alpha|\le 2} \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)^2 G(x)^{-1} D^\alpha h(x,y)\right| + \sum_{k=0}^{2} \left\| \frac{1}{G} D_x^k h \right\|_{\beta+1-k/2}, \tag{57}$$

and

$$||f||_{\tilde{Z}} := \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)G(x)^{-1} f(x,y)\right| + \left\|\frac{1}{G} f\right\|_\beta,$$

where we recall definition (33), and $\beta > \frac{13}{4}$.

Let us make some comments on how the proof is modified. One can readily check that $\tilde{Y}, \tilde{Z}$ are Banach, as in subsection 4.1. Also, since $G \ge \max(|\theta|, \theta^2)$, the map $F : \tilde{Y} \to \tilde{Z}$ in (39) is well-defined, and its continuity, differentiability are still valid, with $D_h F(0,0) = L$ given by (40). To conclude, we need to prove that, for a fixed $f \in \tilde{Z}$, there exists a unique $h \in \tilde{Y}$ such that $Lh = f$. Following the same procedure as in subsection 4.2, we obtain that the h_i's are necessarily given by (46) and (47). We claim (see below) that h defined by (48) does belong to $\tilde{Y}$. Then we conclude the proof by applying Theorem 7 in the same way as in subsection 4.3.

Let us show that h defined by (48) belongs to $\tilde{Y}$. Notice that since $f \in \tilde{Z}$ and (55) holds, we obtain, for all $i \geq 1$,

$$|h_i(x)| = |\rho_i * f_i(x)| \leq \frac{||f||_{\tilde{Z}}}{(1+i)^\beta}(\rho_i * G)(x) \leq \frac{\sigma ||f||_{\tilde{Z}}}{(1+i)^{\beta+1}} G(x), \tag{58}$$

and similarly, since $\lambda_i - \lambda_0 = 2iA$ for all $i \geq 1$, we have

$$h_i'(x) = \sqrt{\lambda_i - \lambda_0}(\rho_i * f_i)(x) \Rightarrow |h_i'(x)| \leq \frac{\sqrt{2A}\sigma ||f||_{\tilde{Z}}}{(1+i)^{\beta+1/2}} G(x),$$
$$h_i''(x) = (\lambda_i - \lambda_0)(\rho_i * f_i)(x) \Rightarrow |h_i''(x)| \leq \frac{2A\sigma ||f||_{\tilde{Z}}}{(1+i)^{\beta}} G(x).$$

The bounds on h_0, h_0', h_0'', can then be deduced. Indeed, from (58) we have

$$\left| f_0(x) + \eta \sum_{i=1}^{+\infty} m_i h_i(x) \right| \leq \left(1 + \eta\sigma \sum_{i=1}^{+\infty} \frac{|m_i|}{(1+i)^{\beta+1}} \right) ||f||_{\tilde{Z}} G(x),$$

where the series converges from (24) and $\beta > \frac{13}{4} > \frac{1}{4}$. Combining this with (47) yields that (58) also holds for $i = 0$. For h_0', h_0'' we proceed as above and thus deduce that $\sum_{k=0}^2 ||G^{-1} D_x^k h||_{\beta+1-k/2} < +\infty$. It remains to prove the upper bound on $|D^\alpha h(x,y)|$ for $|\alpha| \leq 2$. As in subsection 4.2, we have that $h \in C^2(\mathbb{R}^2)$ and (52) holds. Additionally, combining Lemma 9 and Lemma 10, for any $p, q \in \mathbb{N}$ such that $p + q \leq 2$, there holds

$$\begin{aligned} \left|(1+y^2)^2 D_x^p D_y^q h(x,y)\right| &\leq \sum_{i=0}^{+\infty} |h_i^{(p)}(x)| \times |\left(1+y^2\right)^2 \Gamma_i^{(q)}(y)| \\ &\leq CG(x) \sum_{i=0}^{+\infty} \frac{1}{(1+i)^{\beta+1-p/2}} \times i^2 i^{1/4+q/2}, \end{aligned}$$

for some constant $C > 0$. The series converges since $\beta > 13/4$. Hence $h \in \tilde{Y}$. □

It remains to prove Lemma 14.

Proof of Lemma 14. Set

$$0 < \alpha_i := \begin{cases} \sqrt{\lambda_i - \lambda_0} = \sqrt{2iA} & i \geq 1, \\ \sqrt{-\lambda_0} = \sqrt{1-A} & i = 0, \end{cases}$$

and $\tilde{\alpha} := \frac{1}{2}\min(\alpha_0, \alpha_1) > 0$. Notice that $\tilde{\alpha} \leq \frac{1}{2}\alpha_i$ for all $i \in \mathbb{N}$. Define

$$\tilde{G}(x) := \sup_{|t| \geq |x|} \max\left(|\theta(t)|, \theta^2(t), e^{-\frac{\tilde{\alpha}}{2}|t|}\right),$$

which clearly satisfies (54). Then the function

$$G(x) := \begin{cases} \tilde{G}(0) & 0 \leq |x| < 1, \\ \max\left(\frac{\tilde{G}(0)}{2}, \tilde{G}(1)\right) & 1 \leq |x| < 2, \\ \max\left(\frac{\tilde{G}(0)}{2^{k+2}}, \frac{\tilde{G}(1)}{2^{k+1}}, \frac{\tilde{G}(2)}{2^k}, \cdots, \frac{\tilde{G}(2^k)}{2}, \tilde{G}(2^{k+1})\right) & 2^{k+1} \leq |x| < 2^{k+2}, \quad k \in \mathbb{N}, \end{cases}$$

is piecewise constant, satisfies (54) as well as

$$G(x)e^{\tilde{\alpha}x} \to +\infty, \qquad \text{as } x \to +\infty, \tag{59}$$

and

$$G(x/2) \leq 2G(x), \quad \forall x \geq 0. \tag{60}$$

It remains to prove (55). Since both G and $\rho_i * G$ are even, it suffices to consider $x \geq 0$. We write

$$
\begin{aligned}
I(x) &:= \frac{\alpha_i^2}{G(x)}(\rho_i * G)(x) \\
&= \frac{\alpha_i}{2G(x)}\left(\int_{-\infty}^{x/2} e^{-\alpha_i(x-z)}G(z)dz + \int_{x/2}^{x} e^{-\alpha_i(x-z)}G(z)dz + \int_{x}^{+\infty} e^{\alpha_i(x-z)}G(z)dz\right) \\
&=: I_-(x) + I_0(x) + I_+(x).
\end{aligned}
$$

Since G is nonincreasing on $[0,+\infty)$ and satisfies (60), there holds

$$
\begin{aligned}
I_-(x) &\leq \frac{\alpha_i}{2G(x)e^{\alpha_i x}}||G||_\infty \int_{-\infty}^{x/2} e^{\alpha_i z}dz \leq \frac{||G||_\infty}{2G(x)e^{\tilde{\alpha}x}}, \\
I_+(x) &\leq \frac{\alpha_i}{2}e^{\alpha_i x}\int_{x}^{+\infty} e^{-\alpha_i z}dz = \frac{1}{2}, \\
I_0(x) &= \frac{\alpha_i}{2G(x)e^{\alpha_i x}}\int_{x/2}^{x} e^{\alpha_i z}G(z)dz \leq \frac{\alpha_i}{e^{\alpha_i x}}\int_{x/2}^{x} e^{\alpha_i z}dz \leq 1.
\end{aligned}
$$

Since G satisfies (59), $I(x)$ is uniformly bounded on $[0,+\infty)$ independently of $i \in \mathbb{N}$. Consequently, since $\alpha_i^2 = 2iA$ for $i \geq 1$, we see that (55) holds for some $\sigma > 0$. □

4.5 Positivity and control on the y-tails in the periodic and localized cases

In this subsection, we prove estimates (13) and (15), thus completing the proof of Theorem 4 and Theorem 5.

Proof of (13) and (15). We assume either $\theta \in C^L_{per}(\mathbb{R})$ for some $L > 0$ (periodic case), or $\theta \in C_0(\mathbb{R})$ (localized case). From subsection 4.4, in the periodic case, $n^\varepsilon \in Y$ is L-periodic in x, while in the localized case, we have $n^\varepsilon - n^0 \in \tilde{Y}$ where $\tilde{Y}$ is given by (56). Notice that, in both cases, $n^\varepsilon - n^0 \to 0$ as $\varepsilon \to 0$ (in Y or in $\tilde{Y}$ respectively). As a result, by reducing $\varepsilon_0 > 0$ if necessary, there holds that, for any $|\varepsilon| < \varepsilon_0$,

$$
|n^\varepsilon(x,y)| \leq \frac{||n^\varepsilon||_Y}{(1+y^2)^2} \leq \frac{2||n^0||_Y}{(1+y^2)^2}, \qquad \text{in the periodic case,} \tag{61}
$$

$$
\left|n^\varepsilon(x,y) - n^0(y)\right| \leq \frac{||n^\varepsilon - n^0||_{\tilde{Y}}G(x)}{(1+y^2)^2} \leq \frac{G(x)}{(1+y^2)^2}, \qquad \text{in the localized case.} \tag{62}
$$

Assume by contradiction that there is a sequence $\varepsilon_p \to 0$ with $p \geq 1$ such that n^{ε_p} is *not* nonnegative on $\mathbb{R}^2$.

Step 1: n^{ε_p} admits a minimum. Set $m_p := \inf_{(x,y)\in\mathbb{R}^2} n^{\varepsilon_p}(x,y) < 0$, and consider a sequence $(x_p^k, y_p^k)_{k\in\mathbb{N}}$ such that $n^{\varepsilon_p}(x_p^k, y_p^k) \to m_p$ as $k \to +\infty$. From (61)—(62), $n^{\varepsilon_p}(x,y)$ tends to zero as $|y| \to +\infty$ uniformly in $x \in \mathbb{R}$. Thus there exists $Y_p > 0$ such that, for all k, $|y_p^k| \leq Y_p$. Notice that, despite (61)—(62), Y_p depends a priori on p through the value of m_p. On the other hand, in the periodic case, we may consider that $x_p^k \in [0, L]$ while, in the localized case, from (62) we have in the same way $|x_p^k| \leq X_p$. Therefore, assuming $X_p \geq L$, we have in both cases

$$
(x_p^k, y_p^k) \in [-X_p, X_p] \times [-Y_p, Y_p], \quad \forall k \in \mathbb{N}.
$$

Hence, up to a subsequence, (x_p^k, y_p^k) converges to a point $(x_p, y_p) \in [-X_p, X_p] \times [-Y_p, Y_p]$, where n^{ε_p} is thus reaching its minimum.

Step 2: bound on y_p that is uniform w.r.t. p. From the steady state equation (7) for n^{ε_p} evaluated at the minimum point (x_p, y_p), we obtain (recall $m_p < 0$)

$$
\begin{aligned}
0 &= \frac{1}{m_p}\left(\Delta_{x,y} n^{\varepsilon_p}(x_p,y_p) + n^{\varepsilon_p}(x_p,y_p)\left(1 - A^2(y_p - \varepsilon_p\theta(x_p))^2 - \int_{\mathbb{R}} n^{\varepsilon_p}(x_p,y')dy'\right)\right)\\
&\leq 1 - A^2(y_p - \varepsilon_p\theta(x_p))^2 - \int_{\mathbb{R}} n^{\varepsilon_p}(x_p,y')dy'\\
&\leq 1 - A^2y_p^2 + 2A^2\varepsilon_0|y_p|.||\theta||_\infty + A^2\varepsilon_0^2||\theta||_\infty^2 + \begin{cases}\pi||n^0||_Y & \text{in the periodic case,}\\ -\lambda_0 + \frac{\pi}{2}||G||_\infty & \text{in the localized case,}\end{cases}
\end{aligned} \tag{63}
$$

where we used (61)—(62) in the last inequality. The above enforces the existence of some $M > 0$ (independent of p) such that $|y_p| \leq M$.

Step 3: bound on x_p that is uniform w.r.t. p. In the periodic case, this is obvious since we can assume $x_p^k \in [0, L]$. In the localised case, thanks to (62), we have for all $x \in \mathbb{R}$ and $|y| \leq M$,

$$n^{\varepsilon_p}(x,y) \geq n^0(y) - \frac{G(x)}{(1+y^2)^2}||n^{\varepsilon_p} - n^0||_{\tilde{Y}} \geq n^0(M) - G(x).$$

This implies the existence of $X > 0$ independent of p such that $n^{\varepsilon_p}(x,y) \geq \frac{1}{2}n^0(M) > 0$ for any $|x| \geq X$ and $|y| \leq M$. Consequently, for k large enough, we have $|x_p^k| \leq X$. Assuming $X > L$, we thus have in both cases $|x_p| \leq X$.

Step 4: deriving a contradiction. From the above, we can assert that $(x_p, y_p) \in [-X, X] \times [-M, M]$ for p large enough. However, let us underline that $n^0 > 0$ on $\mathbb{R}^2$ and, in both the periodic and the localized case,

$$||n^\varepsilon - n^0||_{L^\infty(\mathbb{R}^2)} \to 0, \quad \text{as } \varepsilon \to 0.$$

As a consequence, for p large enough, there holds $n^{\varepsilon_p} > 0$ on $[-X, X] \times [-M, M]$, which contradicts $m_p = n^{\varepsilon_p}(x_p, y_p) < 0$.

Therefore, by reducing $\varepsilon_0 > 0$ if necessary, we have that, for all $|\varepsilon| \leq \varepsilon_0$, the steady state n^ε is nonnegative. Now, as already seen in (63), there is $C > 0$ such that, for all $x \in \mathbb{R}$, $\int_{\mathbb{R}} n^\varepsilon(x,y)dy \leq C$. We thus deduce from (7) that $-n^\varepsilon_{xx} - n^\varepsilon_{yy} - n^\varepsilon\left[1 - A^2(y - \varepsilon\theta(x))^2 - C\right] \geq 0$. The maximum principle then implies

$$\forall |\varepsilon| \leq \varepsilon_0, \forall (x,y) \in \mathbb{R}^2, \quad n^\varepsilon(x,y) > 0.$$

Last, we prove the exponential control appearing in (13) and (15). Let $a > 0$ be given. Set

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 : 1 - A^2(y - \varepsilon\theta(x))^2 > -a^2\right\}.$$

From (61) and (62), there is $N > 0$ such that $0 < n^\varepsilon \leq N$. We now define

$$\overline{n}(x,y) := Ne^{ay_0}e^{-a|y|}, \quad y_0 := \varepsilon_0||\theta||_\infty + \frac{1}{A}\sqrt{1 + a^2} > 0,$$

so that $\overline{n} \geq n^\varepsilon$ in $\overline{\Omega}$. It remains to prove that $\overline{n} \geq n^\varepsilon$ in Ω^c. Notice that, since $n^\varepsilon \geq 0$ solves (7), there holds

$$\mathcal{E}n^\varepsilon := -n^\varepsilon_{xx} - n^\varepsilon_{yy} - n^\varepsilon\left[1 - A^2(y - \varepsilon\theta(x))^2\right] \leq 0.$$

Meanwhile, in Ω^c,

$$\mathcal{E}\overline{n} = -a^2\overline{n} - \overline{n}\left[1 - A^2(y - \varepsilon\theta(x))^2\right] \geq 0.$$

Due to the maximum principle, we deduce that $n^\varepsilon \leq \overline{n}$ on Ω^c, and thus on $\mathbb{R}^2$. This concludes the proof of (13) and (15). □

5 Construction of pulsating fronts

In this section, we prove Theorem 6 on pulsating fronts.

Let $\varepsilon_0 > 0$ be as in Theorem 4 and, for $|\varepsilon| < \varepsilon_0$, let $n^\varepsilon = n^\varepsilon(x, y)$ be the periodic positive steady state provided by Theorem 4. Let us fix a speed $c_0 \geq c_0^* = 2\sqrt{-\lambda_0}$ and recall that $U = U(z)$ denotes the Fisher-KPP front given by (16) and traveling at speed c_0. We look after a pulsating front solution to (5) in the perturbative form

$$u^\varepsilon(z, x, y) = U(z)n^\varepsilon(x, y) + v_\varepsilon(z, x, y), \quad c_\varepsilon = c_0 + s_\varepsilon,$$

where we understand $z = x - c_\varepsilon t$, meaning that the front spreads at the perturbed speed $c_\varepsilon = c_0 + s_\varepsilon$. Plugging this into (5), using the steady state equation (7) for $n^\varepsilon(x, y)$ and the front equation (16) for $U(z)$, we are left to find $(s_\varepsilon, v_\varepsilon)$ satisfying $\mathcal{F}(\varepsilon, s_\varepsilon, v_\varepsilon) = 0$ where

$$\begin{aligned}\mathcal{F}(\varepsilon, s, v) := {} & v_{zz} + 2v_{xz} + v_{xx} + v_{yy} + (c_0 + s)v_z + sU'(z)n^\varepsilon(x, y) + 2U'(z)n_x^\varepsilon(x, y) \\ & + v\left(1 - A^2(y - \varepsilon\theta(x))^2 - U(z)\int_{\mathbb{R}} n^\varepsilon(x, y')dy' - \int_{\mathbb{R}} v(z, x, y')dy'\right) \\ & - U(z)n^\varepsilon(x, y)\int_{\mathbb{R}} v(z, x, y')dy' + U(z)(1 - U(z))n^\varepsilon(x, y)\left(\lambda_0 + \int_{\mathbb{R}} n^\varepsilon(x, y')dy'\right). \end{aligned} \tag{64}$$

However, since the elliptic operator appearing in the right-hand side above is degenerate in the (z, x) variables, we need to consider the regularization

$$\mathcal{F}^\mu(\varepsilon, s, v) := \mathcal{F}(\varepsilon, s, v) + \mu v_{xx}, \quad 0 < \mu \ll 1. \tag{65}$$

To prove Theorem 6, the very crude strategy is as follows. We first apply the Implicit Function Theorem, namely Theorem 7, to $\mathcal{F}^\mu \colon \mathbb{R} \times \mathbb{R} \times \mathcal{Y}_\mu \to \mathcal{Z}$ where the function spaces $\mathcal{Y}_\mu$ and $\mathcal{Z}$ are appropriately chosen. This will provide a couple $(s_{\varepsilon,\mu}, v_{\varepsilon,\mu}) \in \mathbb{R} \times \mathcal{Y}_\mu$ for any $\mu > 0$ small enough. Then, we shall obtain $s_\varepsilon, v_\varepsilon$ by passing to the limit $\mu \to 0$. See Remark in subsection 5.1 for more details on the key ideas of the proof.

By assumption, see Theorem 6, there are $\gamma > 3$, $k \geq 0$ and $0 \leq \delta < 1$ with $k + \delta > \gamma + \frac{1}{2}$ such that θ belongs to $C^{k,\delta}(\mathbb{R}) \cap C^L_{per}(\mathbb{R})$, and so does θ^2. In particular, the Fourier coefficients of θ and θ^2 decay at least at speed $|m|^{-(k+\delta)}$ as $|m| \to \infty$, that is

$$\exists K_\theta > 0, \forall m \in \mathbb{Z}, \ \max\left(|\theta_m|, |(\theta^2)_m|\right) \leq \frac{K_\theta}{(1 + |m|)^{k+\delta}}, \tag{66}$$

where we denote

$$\theta_m := \frac{1}{L}\int_0^L \theta(x)e^{-\frac{2i\pi mx}{L}}dx, \qquad (\theta^2)_m := \frac{1}{L}\int_0^L \theta^2(x)e^{-\frac{2i\pi mx}{L}}dx.$$

5.1 Function spaces

We first present a few notations that will be used below. For any function $f = f(z, x, y) \in C_b(\mathbb{R}^3)$ such that $f(z, x, \cdot) \in L^2(\mathbb{R})$ and $f(z, x + L, y) = f(z, x, y)$ for all z, x, y, we denote

$$f_j(z, x) := \int_{\mathbb{R}} f(z, x, y)\Gamma_j(y)dy, \tag{67}$$

that is f_j denotes the j-th coordinate of f along the basis of eigenfunctions $(\Gamma_j = \Gamma_j(y))_{j\in\mathbb{N}}$. We also define

$$f_j^n(z) := \frac{1}{L}\int_0^L f_j(z, x)e^{-\frac{2i\pi n}{L}x}dx = \frac{1}{L}\int_0^L f_j(z, x)e_{-n}(x)dx, \tag{68}$$

$$e_n(x) := e^{\frac{2i\pi n}{L}x} = e^{i\sigma nx}, \qquad \sigma := \frac{2\pi}{L}, \ n \in \mathbb{Z}, \tag{69}$$

that is $f_j^n(z)$ denotes the n-th Fourier coefficient of $x \mapsto f_j(z,x)$.

Now, for a $\kappa \in \left(0, -\frac{1}{2}c_0 + \frac{1}{2}\sqrt{c_0^2 - 4\lambda_0}\right)$ to be precised later, we define

$$\mathcal{Y}_\mu := \left\{ v \in C^2(\mathbb{R}^3) \;\middle|\; \begin{array}{c} v(z,x+L,y) = v(z,x,y) \quad \text{on } \mathbb{R}^3, \\ \exists C>0,\ \forall |\alpha| \le 2, \quad |D^\alpha v(z,x,y)| \le \frac{Ce^{-\kappa|z|}}{(1+y^2)^2} \quad \text{on } \mathbb{R}^3, \\ \exists K>0,\ \forall n \in \mathbb{Z},\ \forall j \in \mathbb{N},\ \forall k \le 2, \quad \text{there holds} \\ |(v_j^n)^{(k)}(z)| \le \frac{Ke^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1+|n|^k + j^{k/2}}{1+\mu n^2 + j + |n|} \quad \text{on } \mathbb{R} \end{array} \right\}, \tag{70}$$

$$\mathcal{Z} := \left\{ f \in C(\mathbb{R}^3) \;\middle|\; \begin{array}{c} f(z,x+L,y) = f(z,x,y) \quad \text{on } \mathbb{R}^3, \\ \exists C>0, \quad |f(z,x,y)| \le \frac{Ce^{-\kappa|z|}}{1+y^2} \quad \text{on } \mathbb{R}^3, \\ \exists K>0,\ \forall n \in \mathbb{Z},\ \forall j \in \mathbb{N}, \quad \text{there holds} \\ |f_j^n(z)| \le \frac{Ke^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \quad \text{on } \mathbb{R} \end{array} \right\}. \tag{71}$$

Obviously those spaces also depend on parameters κ, β and γ which will be fixed later, and we therefore only indicate the dependence on μ. We equip the space $\mathcal{Y}_\mu$ with the norm

$$||v||_{\mathcal{Y}_\mu} := \sum_{|\alpha| \le 2} \sup_{(z,x,y)\in\mathbb{R}^3} \left[\left|(1+y^2)^2 D^\alpha v(z,x,y)\right| e^{\kappa|z|} \right] + ||v||_{\beta,\gamma,\mu}, \tag{72}$$

where

$$||v||_{\beta,\gamma,\mu} := \sum_{k=0}^{2} \sup_{n\in\mathbb{Z}, j\in\mathbb{N}} \left[(1+j)^\beta (1+|n|)^\gamma \frac{1+\mu n^2 + j + |n|}{1+|n|^k + j^{k/2}} \sup_{z\in\mathbb{R}} \left| (v_j^n)^{(k)}(z) e^{\kappa|z|} \right| \right].$$

We equip the space $\mathcal{Z}$ with the norm

$$||f||_{\mathcal{Z}} = \sup_{(z,x,y)\in\mathbb{R}^3} \left[\left|(1+y^2) f(z,x,y)\right| e^{\kappa|z|} \right] + ||f||_{\beta,\gamma}, \tag{73}$$

where

$$||f||_{\beta,\gamma} := \sup_{n\in\mathbb{Z}, j\in\mathbb{N}} \left[(1+j)^\beta (1+|n|)^\gamma \sup_{z\in\mathbb{R}} \left| f_j^n(z) e^{\kappa|z|} \right| \right].$$

Remark 15 (Choice of the function spaces and overview of the proof of Theorem 6)**.** *Let us comment on the spaces $\mathcal{Y}_\mu, \mathcal{Z}$ and the two controls appearing in their definition. As in the stationary case, i.e. Section 4, the crux of the proof is to show that $\mathcal{L}^\mu := D_{(s,v)}\mathcal{F}^\mu(0,0,0)$, given by (95), is bijective from $\mathbb{R} \times \mathcal{S}_\mu$ to $\mathcal{Z}$, where $\mathcal{S}_\mu \subset \mathcal{Y}_\mu$ is to be determined, that is for every fixed $f \in \mathcal{Z}$, there is a unique $s_\mu \in \mathbb{R}$ and a unique $v_\mu \in \mathcal{S}_\mu$ such that $\mathcal{L}^\mu(s_\mu, v_\mu) = f$. Using the controls on the y-tails provided by the first constraint in the definition of $\mathcal{Y}_\mu$ and $\mathcal{Z}$, and then the L-periodicity in x, we decompose successively v_μ and f along the eigenfunction bases $(\Gamma_j)_{j\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{Z}}$ respectively, leading to (100), where we denoted $v = v_\mu$ to ease readability. Next, the control on f_j^n, i.e. the second constraint in the definition of $\mathcal{Z}$, allows to prove the bound (128).*

However, the operators $\mathcal{L}_{n,j,\mu}$ defined by (100), (102) and (103) might not be injective. Thus, in order to ensure the uniqueness, we require that $(v_\mu)_j^n$ belongs to a subspace $\mathcal{S}_{n,j,\mu}$ of the departure space of $\mathcal{L}_{n,j,\mu}$. These additional conditions lead to $v_\mu \in \mathcal{S}_\mu$, after reconstruction of v_μ according to (138). To show that $v_\mu \in C_b^2(\mathbb{R}^3)$ and v_μ satisfies the first control in $\mathcal{Y}_\mu$, we require $\beta > \frac{17}{4}$ and $\gamma > 2$. This allows to apply Theorem 7 and deduce the existence of $s_{\varepsilon,\mu}, v_{\varepsilon,\mu}$ such that $\mathcal{F}^\mu(\varepsilon, s_{\varepsilon,\mu}, v_{\varepsilon,\mu}) = 0$ for any $|\varepsilon| \le \overline{\varepsilon}_0$.

The next step is to let $\mu \to 0$ in $\mathcal{F}^\mu(\varepsilon, s_{\varepsilon,\mu}, v_{\varepsilon,\mu}) = 0$. However, to do so, we first require that $\overline{\varepsilon}_0$ may be fixed independently on μ. This is actually true from the crucial observation that, despite $\mathcal{F}^\mu$ is unbounded with respect to μ, both $(\mathcal{L}^\mu)^{-1}$ and $\mathcal{F}^\mu - \mathcal{L}^\mu$, see (141), are bounded (see subsection 5.4).

Last, to ensure that a subsequence of $(v_{\varepsilon,\mu})_\mu$ converges as $\mu \to 0$, we need to redo the above proof by replacing $\mathcal{Y}_\mu$ and $\mathcal{Z}$ with (145) and (146) respectively. This allows to obtain C_b^3 regularity for $v_{\varepsilon,\mu}$, at the cost of the assumptions $\beta > \frac{19}{4}$ and $\gamma > 3$. Then, the L-periodicity on x and the controls on the y- and z-tails in the definition of (145) allow in some sense to compactify the domain of definition of $v_{\varepsilon,\mu}$, so that we can adapt the proof of the Arzelà-Ascoli theorem and conclude, see subsection 5.5.

In what follows, we will repeatedly use the following straightforward estimates:

$$\forall v \in \mathcal{Y}_\mu,\ \forall |\alpha| \le 2,\ \forall (z,x,y) \in \mathbb{R}^3, \qquad |D^\alpha v(z,x,y)| \le ||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}}{(1+y^2)^2}, \tag{74}$$

$$\forall v \in \mathcal{Y}_\mu,\ \forall k \le 2,\ \forall (n,j) \in \mathbb{Z}\times\mathbb{N},\ \forall z \in \mathbb{R}, \quad |(v_j^n)^{(k)}(z)| \le ||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}\left(1+|n|^k + j^{k/2}\right)}{(1+j)^\beta (1+|n|)^\gamma \left(1+\mu n^2 + j + |n|\right)}, \tag{75}$$

$$\forall f \in \mathcal{Z},\ \forall (z,x,y) \in \mathbb{R}^3, \qquad |f(z,x,y)| \le ||f||_{\mathcal{Z}} \frac{e^{-\kappa|z|}}{1+y^2}, \tag{76}$$

$$\forall f \in \mathcal{Z},\ \forall (n,j) \in \mathbb{Z}\times\mathbb{N},\ \forall z \in \mathbb{R}, \qquad |f_j^n(z)| \le ||f||_{\mathcal{Z}} \frac{e^{-\kappa|z|}}{(1+j)^\beta (1+|n|)^\gamma}, \tag{77}$$

$$\forall v \in \mathcal{Y}_\mu,\ \forall (z,x) \in \mathbb{R}^2, \qquad \left|\int_{\mathbb{R}} v(z,x,y)dy\right| \le \frac{\pi}{2}||v||_{\mathcal{Y}_\mu} e^{-\kappa|z|}. \tag{78}$$

Also, we claim that there exists $K_A > 0$, that depends only on A, such that

$$\forall v \in \mathcal{Y}_\mu,\ \forall z \in \mathbb{R},\ \forall n \in \mathbb{Z}, \qquad \left|\frac{1}{L}\int_0^L \int_{\mathbb{R}} v(z,x,y)dy e_{-n}(x)dx\right| \le K_A ||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}}{(1+|n|)^{\gamma+1}}. \tag{79}$$

Indeed, from (75), we have (as usual the constant $C > 0$ is independent of z, x, y, j and n but may change from line to line)

$$\left|v_j^n(z)\right| \le C||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}}{(1+j)^\beta (1+|n|)^{\gamma+1}}.$$

Let us recall that $\beta > \frac{19}{4}$ and $\gamma > 3$. Thus we obtain

$$|v_j(z,x)| \le \sum_{n=-\infty}^{\infty} |v_j^n(z)| \le C||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}}{(1+j)^\beta}.$$

Since (24) holds, we deduce

$$\int_{\mathbb{R}} v(z,x,y)dy = \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{N}} v_j(z,x)\Gamma_j(y)\right) dy = \sum_{j\in\mathbb{N}} m_j v_j(z,x),$$

where we recall the notation $m_j := \int_{\mathbb{R}} \Gamma_j(y)dy$. This in turn leads to

$$\left|\frac{1}{L}\int_0^L \left(\int_{\mathbb{R}} v(z,x,y)dy\right) e_{-n}(x)dx\right| = \left|\sum_{j\in\mathbb{N}} m_j v_j^n(z)\right| \le C||v||_{\mathcal{Y}_\mu} \frac{e^{-\kappa|z|}}{(1+|n|)^{\gamma+1}} \sum_{j\in\mathbb{N}} \frac{|m_j|}{(1+j)^\beta},$$

which proves the claim (79) using again (24).

Lemma 16 ($\mathcal{Y}_\mu$ and $\mathcal{Z}$ are Banach)**.** *For all $0 < \mu < 1$, the spaces $\mathcal{Y}_\mu$ given by (70), and $\mathcal{Z}$ given by (71), are Banach spaces when equipped with their respective norm $||.||_{\mathcal{Y}_\mu}$ given by (72), and $||.||_{\mathcal{Z}}$ given by (73).*

Proof. Let us fix $0 < \mu < 1$. For the sake of completeness, we give a short proof that $\mathcal{Y}_\mu$ is Banach, the proof for $\mathcal{Z}$ being similar. Let $(v_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{Y}_\mu$. Since the embedding $\mathcal{Y}_\mu \hookrightarrow C_b^2(\mathbb{R}^3)$ is continuous and $C_b^2(\mathbb{R}^3)$ is Banach, there is $v \in C_b^2(\mathbb{R}^3)$ such that $||v_m - v||_{C_b^2(\mathbb{R}^3)} \to 0$.

Let us prove that $v \in \mathcal{Y}_\mu$. The L-periodicity in x of v is obvious. Following the same arguments as in the proof of Lemma 12, there exists $C > 0$ such that

$$|D^\alpha v(z,x,y)| \leq \frac{Ce^{-\kappa|z|}}{(1+y^2)^2}, \tag{80}$$

for all $|\alpha| \leq 2$ and $(z,x,y) \in \mathbb{R}^3$. Next, similarly to the proof of Lemma 12, the sequence $K_m := \sum_{k=0}^2 ||v_m||_{\gamma,\beta,\mu}$ is bounded for all $m \in \mathbb{N}$ by some $K > 0$. Since (80) holds, we deduce by the dominated convergence theorem that for any $k \leq 2$,

$$\begin{aligned}\left|(v_j^n)^{(k)}(z)\right| &= \lim_{m\to\infty} \left|\frac{1}{L}\int_0^L \left(\int_{\mathbb{R}} D_z^k v_m(z,x,y)\Gamma_j(y)dy\right) e_{-n}(x)dx\right| \\ &\leq \frac{Ke^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1+|n|^k+j^{k/2}}{1+\mu n^2+j+|n|}.\end{aligned}$$

Therefore $v \in \mathcal{Y}_\mu$. From there, classical arguments (that we omit) yield that $||v_m - v||_{\mathcal{Y}_\mu} \to 0$. □

We now state some preliminary results. For better readability, we denote yv the function $(z,x,y) \mapsto yv(z,x,y)$, and similarly for $y^2 v$.

Lemma 17 (Controlling in $\mathcal{Z}$ the v terms of $\mathcal{F}^\mu(\varepsilon, s, v)$)**.** *There exists $C > 0$ such that, for any $\mu \in (0,1)$ and $v = v(z,x,y) \in \mathcal{Y}_\mu$,*

$$\max\left(||v||_{\mathcal{Z}}, ||D_z v||_{\mathcal{Z}}, ||yv||_{\mathcal{Z}}, ||y^2 v||_{\mathcal{Z}}\right) \leq C||v||_{\mathcal{Y}_\mu}, \tag{81}$$

$$\max_{|\alpha|\leq 2} ||D^\alpha v||_{\mathcal{Z}} \leq \mu^{-1} C ||v||_{\mathcal{Y}_\mu}. \tag{82}$$

Also, set $\rho \geq 0$ and assume $b = b(z,x) \in C_b(\mathbb{R}^2)$ satisfies

$$\begin{cases} b(z,x+L) = b(z,x) & \forall z,x \in \mathbb{R}, \\ |b_m(z)| := \left|\frac{1}{L}\int_0^L b(z,x)e_{-m}(x)dx\right| \leq \frac{K_b}{(1+|m|)^{\gamma+\rho}} & \forall m \in \mathbb{Z},\ \forall z \in \mathbb{R}, \end{cases} \tag{83}$$

for some $K_b > 0$. Then there are $C_\rho, C'_\rho > 0$ such that, for any $\mu \in (0,1)$ and $v = v(z,x,y) \in \mathcal{Y}_\mu$,

$$||bv||_{\mathcal{Z}} \leq \begin{cases} (\mu^{-1}C_\rho K_b + ||b||_{L^\infty(\mathbb{R}^2)})||v||_{\mathcal{Y}_\mu} & \text{if } \rho = 0, \\ (C_\rho K_b + ||b||_{L^\infty(\mathbb{R}^2)})||v||_{\mathcal{Y}_\mu} & \text{if } \rho > 0, \end{cases} \tag{84}$$

and

$$||byv||_{\mathcal{Z}} \leq (C'_\rho K_b + ||b||_{L^\infty(\mathbb{R}^2)})||v||_{\mathcal{Y}_\mu} \qquad \text{if } \rho > 1/2. \tag{85}$$

Proof. Fix $\mu \in (0,1)$. By definition of $\mathcal{Y}_\mu$, for any function $w \in \{D^\alpha v, yv, y^2 v, bv, byv\}$, it is clear that w is L-periodic in x and satisfies, thanks to (74),

$$|w(z,x,y)| \leq \frac{C||v||_{\mathcal{Y}_\mu} e^{-\kappa|z|}}{1+y^2}, \qquad \forall (z,x,y) \in \mathbb{R}^3,$$

with $C = ||b||_{L^\infty(\mathbb{R}^2)}$ if $w \in \{bv, byv\}$, and $C = 1$ otherwise. Thus in order to prove (81) and (82), it is enough to control $||w||_{\beta,\gamma}$ for each $w \in \{D^\alpha v, yv, y^2 v, bv, byv\}$.

If $w = yv$, by virtue of (22), we have

$$(yv)_j^n(z) = \frac{1}{L}\int_0^L \left(\int_{\mathbb{R}} v(z,x,y) y \Gamma_j(y) dy\right) e_{-n}(x)dx = \begin{cases} p_j^+ v_{j+1}^n(z) + p_j^- v_{j-1}^n(z) & j \geq 1, \\ p_0^+ v_1^n(z) & j = 0. \end{cases}$$

From (75), we thus obtain $||yv||_{\beta,\gamma} \leq C||v||_{\mathcal{Y}_\mu}$ for some $C > 0$. One can readily check that the same is true for $w = y^2 v$.

Now, set $|\alpha| \leq 2$ and $w = D^\alpha v$. If $w = v$ or $w = D_z v$, then from (75), we deduce $||w||_{\beta,\gamma} \leq ||v||_{\beta,\gamma,\mu} \leq ||v||_{\mathcal{Y}_\mu}$. If $w = D_z^2 v$, then (75) yields $||D_z^2 v||_{\beta,\gamma} \leq \mu^{-1}||v||_{\mathcal{Y}_\mu}$ since $0 < \mu < 1$. Now, consider $w = D_x^k v$ with $k \in \{1, 2\}$. Then by integration by parts there holds

$$(D_x^k v)_j^n(z) = \frac{1}{L}\int_0^L D_x^k \left(\int_{\mathbb{R}} v(z,x,y)\Gamma_j(y)dy\right) e_{-n}(x)dx = (in\sigma)^k\, v_j^n(z),$$

which, thanks to (75) implies $||D_x v||_{\beta,\gamma} \leq \sigma||v||_{\mathcal{Y}_\mu}$ and $||D_x^2 v||_{\beta,\gamma} \leq \mu^{-1}\sigma||v||_{\mathcal{Y}_\mu}$. As for $w = D_y^k v$ with $k \leq 2$, the proof is similar to $w = y^k v$ by using (23) instead of (22). Therefore (82) holds for $D^\alpha \in \{D_z^k, D_x^k, D_y^k\}$ with $0 \leq k \leq 2$. The proof for the cross derivatives $D^\alpha \in \{D_{xy}, D_{xz}, D_{yz}\}$ results from a combination of the above arguments. Therefore we proved (81)—(82).

Next, let us consider $w = bv$. Since (83) holds with $\gamma + \rho \geq \gamma > 3 > 1$, the Fourier series of $b(z, \cdot)$ converges uniformly on $\mathbb{R}$ and we have pointwise

$$b(z,x) = \sum_{m=-\infty}^{\infty} b_m(z)e_m(x).$$

This leads to

$$\begin{aligned}
(bv)_j^n(z) &= \frac{1}{L}\int_0^L b(z,x)\left(\int_{\mathbb{R}} v(z,x,y)\Gamma_j(y)dy\right) e_{-n}(x)dx \\
&= \frac{1}{L}\int_0^L \left(\sum_{m=-\infty}^{\infty} b_m(z)e_m(x)\right) v_j(z,x)e_{-n}(x)dx \\
&= \sum_{m=-\infty}^{\infty} b_m(z)v_j^{n-m}(z) = \sum_{m=-\infty}^{\infty} b_{n-m}(z)v_j^m(z).
\end{aligned}$$

Let us first assume that $\rho = 0$. The controls (75) and (83) then yield

$$\begin{aligned}
\left|(bv)_j^n(z)\right| &\leq \sum_{m=-\infty}^{\infty} |b_{n-m}(z)| \times \left|v_j^m(z)\right| \\
&\leq K_b||v||_{\mathcal{Y}_\mu} \sum_{m=-\infty}^{\infty} \frac{1}{(1+|n-m|)^\gamma} \times \frac{e^{-\kappa|z|}}{(1+j)^\beta(1+|m|)^\gamma} \times \frac{1}{1+\mu m^2 + j + |m|}.
\end{aligned}$$

From there, one can readily check, by studying all possible cases on the signs of n, m and $n-m$, that

$$\frac{1}{(1+|n-m|)(1+|m|)} \leq \frac{1}{1+|n|}, \qquad \forall m, n \in \mathbb{Z}.$$

Therefore

$$\begin{aligned}
\left|(bv)_j^n(z)\right| &\leq \frac{K_b||v||_{\mathcal{Y}_\mu}e^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{1+\mu m^2 + j + |m|} \\
&\leq \frac{\mu^{-1}K_b||v||_{\mathcal{Y}_\mu}e^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{1+m^2},
\end{aligned}$$

which gives (84) for $\rho = 0$. Meanwhile, if $\rho > 0$, similar calculations yield

$$\begin{aligned}
\left|(bv)_j^n(z)\right| &\leq \frac{K_b||v||_{\mathcal{Y}_\mu}e^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{(1+|n-m|)^\rho} \times \frac{1}{1+\mu m^2 + j + |m|} \\
&\leq \frac{K_b||v||_{\mathcal{Y}_\mu}e^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{(1+|n-m|)^\rho(1+|m|)}.
\end{aligned}$$

Since $\rho > 0$, Hölder inequality shows that the sum of the infinite series above is bounded by some $C_\rho > 0$ which is independent of $n \in \mathbb{Z}$. This gives (84) for $\rho > 0$.

Finally, let us prove (85). With (22), we obtain that, for some $C_{A,\beta} > 0$,

$$\begin{aligned}\left|(byv)_j^n(z)\right| &\leq \frac{K_b ||v||_{\mathcal{Y}_\mu} e^{-\kappa|z|}}{(1+j)^\beta (1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{(1+|n-m|)^\rho} \times \frac{C_{A,\beta}\sqrt{j}}{1+j+|m|} \\ &\leq \frac{K_b ||v||_{\mathcal{Y}_\mu} e^{-\kappa|z|}}{(1+j)^\beta (1+|n|)^\gamma} \sum_{m=-\infty}^{\infty} \frac{1}{(1+|n-m|)^\rho} \times \frac{C_{A,\beta}}{\sqrt{2}(1+|m|)^{\frac{1}{2}}},\end{aligned}$$

as easily seen by studying the maximum of $j \in [0, +\infty) \mapsto \frac{\sqrt{j}}{1+j+|m|}$. Since $\rho > \frac{1}{2}$, Hölder inequality shows that the sum of the infinite series above is bounded by some $C'_\rho > 0$ which is independent of $n \in \mathbb{Z}$. This gives (85). □

To conclude this subsection, we prove that, by taking $|\varepsilon|$ possibly smaller, we obtain some estimates on the steady state $n^\varepsilon = n^\varepsilon(x,y)$ in the $\mathcal{Y}_\mu, \mathcal{Z}$ norms. For better readability, we denote $e^{-\kappa|z|}h$ the function $(z,x,y) \mapsto e^{-\kappa|z|}h(x,y)$ and similarly for $e^{-\kappa|z|}h_x$.

Lemma 18 (The steady state n^ε when $\theta \in C^L_{per}(\mathbb{R})$ further satisfies (66))**.** *Fix $\beta > \frac{17}{4}$ and $\gamma > 2$. Let the conditions of Theorem 4 hold. Assume further that $\theta \in C^L_{per}(\mathbb{R})$ satisfies (66). Then, there is $\varepsilon_0^* > 0$ such that, for any $|\varepsilon| \leq \varepsilon_0^*$,*

there is a unique $n^\varepsilon \in Y^*$ such that n^ε solves (7),

where the function space Y^* is given by (91). Additionally, we have $||n^\varepsilon - n^0||_{Y^*} \to 0$, as $\varepsilon \to 0$, where $||\cdot||_{Y^*}$ is the norm given by (92). Finally, there are $K_\sigma > 0$ (depending only on $\sigma = \frac{2\pi}{L}$), $K_\kappa > 0$ (depending only on κ) and $K_A > 0$ (depending only on A) such that, for any $h \in Y^*$,

$$||e^{-\kappa|z|}h||_{\mathcal{Z}} \leq ||h||_{Y^*}, \tag{86}$$

$$||e^{-\kappa|z|}h_x||_{\mathcal{Z}} \leq K_\sigma ||h||_{Y^*}, \tag{87}$$

$$||e^{-\kappa|z|}h||_{\mathcal{Y}_\mu} \leq K_\kappa ||h||_{Y^*}, \qquad \forall 0 < \mu < 1, \tag{88}$$

$$\left|\int_{\mathbb{R}} h(x,y)dy\right| \leq \frac{\pi}{2}||h||_{Y^*} \qquad \forall x \in \mathbb{R}, \tag{89}$$

$$\left|\frac{1}{L}\int_0^L \left(\int_{\mathbb{R}} h(x,y)dy\right) e_{-n}(x)dx\right| \leq \frac{K_A ||h||_{Y^*}}{(1+|n|)^{\gamma+2}} \qquad \forall n \in \mathbb{Z}. \tag{90}$$

Proof. In the context of this proof, for any function $h = h(x,y) \in C_b(\mathbb{R}^2)$ such that $h(x,\cdot) \in L^2(\mathbb{R})$ and $h(x+L,y) = h(x,y)$ for all $(x,y) \in \mathbb{R}^2$, we denote

$$h_j(x) := \int_{\mathbb{R}} h(x,y)\Gamma_j(y)dy, \qquad h_j^n := \frac{1}{L}\int_0^L h_j(x)e_{-n}(x)dx,$$

that is h_j^n is the n-th Fourier coefficient of h_j, which is the j-th coordinate of h along the basis of eigenfunctions $(\Gamma_j)_{j\in\mathbb{N}}$. We now define

$$Y^* := \left\{ h \in C^2(\mathbb{R}^2) \;\middle|\; \begin{array}{c} h(x+L,y) = h(x,y), \qquad \forall x,y \in \mathbb{R}, \\ \exists C > 0,\ \forall |\alpha| \leq 2, \quad |D^\alpha h(x,y)| \leq \frac{C}{(1+y^2)^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0,\ \forall n \in \mathbb{Z},\ \forall j \in \mathbb{N}, \quad \left|h_j^n\right| \leq \frac{K}{(1+j)^\beta (1+|n|)^\gamma} \times \frac{1}{1+j+n^2} \end{array} \right\}, \tag{91}$$

$$Z^* := \left\{ f \in C(\mathbb{R}^2) \;\middle|\; \begin{array}{c} f(x+L,y) = f(x,y), \qquad \forall x,y \in \mathbb{R}, \\ \exists C > 0, \quad |f(x,y)| \leq \frac{C}{1+y^2} \quad \text{on } \mathbb{R}^2, \\ \exists K > 0,\ \forall n \in \mathbb{Z},\ \forall j \in \mathbb{N}, \quad \left|f_j^n\right| \leq \frac{K}{(1+j)^\beta (1+|n|)^\gamma} \end{array} \right\},$$

$$||h||_{Y^*} := \sum_{|\alpha|\leq 2} \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)^2 D^\alpha h(x,y)\right| + \sup_{n\in\mathbb{Z}, j\in\mathbb{N}} \left[(1+j)^\beta (1+|n|)^\gamma (1+j+n^2)\left|h_j^n\right|\right], \tag{92}$$

$$||f||_{Z^*} := \sup_{(x,y)\in\mathbb{R}^2} \left|(1+y^2)f(x,y)\right| + \sup_{n\in\mathbb{Z}, j\in\mathbb{N}} \left[(1+j)^\beta (1+|n|)^\gamma \left|f_j^n\right|\right].$$

The proof of Lemma 18 relies on applying the Implicit Function Theorem, namely Theorem 7, to the function $F = F(\varepsilon, h)$ defined by (39). Firstly, adapting the proof of Lemmas 12 and 13, one can readily check that Y^*, Z^* are Banach spaces, that $F : \mathbb{R} \times Y^* \to Z^*$ is well-defined, and that conditions (i)—(ii) of Theorem 7 are satisfied, with $D_h F(0,0) = L$ given by (40). It remains to prove that $L : Y^* \to Z^*$ is bijective. Following the same procedure as in subsection 4.2, we have that h_j, f_j satisfy (44)—(45). We now use the Fourier coefficients: for $n \in \mathbb{Z}$, we multiply equations (44)—(45) by $\frac{1}{L}e_{-n}(x)$ and integrate over $x \in [0, L]$. We obtain

$$\begin{cases} \left(-n^2\sigma^2 - (\lambda_j - \lambda_0)\right) h_j^n = f_j^n, & j \geq 1, \\ \left(-n^2\sigma^2 + \lambda_0\right) h_0^n = f_0^n + \eta \sum_{\ell=1}^{+\infty} m_\ell h_\ell^n, & j = 0. \end{cases} \tag{93}$$

For $j \geq 1$, since $0 < \lambda_j - \lambda_0$, we see that, for any $n \in \mathbb{Z}$, there is a unique $h_j^n \in \mathbb{C}$ solving the first equation in (93). Since $f \in Z^*$, we have

$$|f_j^n| \leq \frac{||f||_{Z^*}}{(1+j)^\beta (1+|n|)^\gamma},$$

which leads to

$$|h_j^n| = \frac{|f_j^n|}{n^2\sigma^2 + 2jA} \leq \frac{K||f||_{Z^*}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1}{1+j+n^2},$$

for some $K = K(A, L) > 0$. From there, in view of (24) and $\beta > \frac{17}{4} > \frac{5}{4}$, the right-hand side of the second equation of (93) is well-defined, and bounded by $M||f||_{Z^*}(1+|n|)^{-\gamma}$ for some $M > 0$ independent of n. Therefore we obtain

$$|h_0^n| \leq \frac{K||f||_{Z^*}}{(1+|n|)^\gamma (1+n^2)},$$

by taking K possibly larger. It remains to reconstruct h and prove that it belongs to Y^*. Since $\gamma > 2 > 1$, we have for any $0 \leq k \leq 2$

$$h_j^{(k)}(x) = \sum_{n\in\mathbb{Z}} h_j^n (i\sigma n)^k e_n(x),$$

from which we deduce

$$||h_j^{(k)}||_\infty \leq \frac{K||f||_{Z^*}}{(1+j)^\beta} \sum_{n\in\mathbb{Z}} \frac{(\sigma|n|)^k}{(1+|n|)^{\gamma+2}} \leq \frac{C||f||_{Z^*}}{(1+j)^\beta},$$

for some $C = C(A, L, \gamma) > 0$. In other words, we obtain the estimates playing the roles of (49)—(51). Then, like the rest of the proof in subsection 4.2, we prove that $h \in Y^*$ since $\beta > \frac{17}{4}$. Thus L is bijective. Finally, we apply Theorem 7, which leads to the existence of $\varepsilon_0^* > 0$ such that, for any $|\varepsilon| \leq \varepsilon_0^*$, there exists a unique function $h^\varepsilon \in Y^*$ such that $F(\varepsilon, h^\varepsilon) = 0$, with $||h^\varepsilon||_{Y^*} \to 0$ as $\varepsilon \to 0$. Since $n^0 \in Y^*$, we deduce that $n^\varepsilon(x,y) = n^0(y) + h^\varepsilon(x,y) \in Y^*$ and n^ε solves (7).

To conclude, (86)—(88) simply follow from the definitions of $\mathcal{Z}, \mathcal{Y}_\mu, Y^*$, given that $(h_x)_j^n = (in\sigma)h_j^n$. Meanwhile, (89)—(90) is proved in the same way as (78)—(79). □

5.2 Checking assumptions (i) and (ii) of Theorem 7

For the rest of this section, we assume that $|\varepsilon| \leq \varepsilon_0^*$, where ε_0^* is obtained from Lemma 18. We also recall that $\mu \in (0,1)$. Equipped with the above spaces $\mathcal{Y}_\mu$ and $\mathcal{Z}$, we thus consider

$$\begin{aligned} \mathcal{F}^\mu(\varepsilon, s, v) = {} & v_{zz} + 2v_{xz} + (1+\mu)v_{xx} + v_{yy} + (c_0+s)v_z + sU'(z)n^\varepsilon(x,y) + 2U'(z)n_x^\varepsilon(x,y) \\ & + v\left(1 - A^2(y-\varepsilon\theta(x))^2 - U(z)\int_{\mathbb{R}} n^\varepsilon(x,y')dy' - \int_{\mathbb{R}} v(z,x,y')dy'\right) \\ & - U(z)n^\varepsilon(x,y)\int_{\mathbb{R}} v(z,x,y')dy' + U(z)(1-U(z))n^\varepsilon(x,y)\left(\lambda_0 + \int_{\mathbb{R}} n^\varepsilon(x,y')dy'\right). \end{aligned}$$

Recall that $n^\varepsilon(x,y) = n^0(y)$ when $\varepsilon = 0$ and $\int_{\mathbb{R}} n^0(y')dy' = -\lambda_0$. Consequently $\mathcal{F}^\mu(0,0,0) = 0$.

Checking assumptions (i) and (ii) of Theorem 7. Fix $\mu \in (0,1)$. We first prove that $\mathcal{F}^\mu \colon \mathcal{Y}_\mu \to \mathcal{Z}$ is well-defined and continuous at $(0,0,0)$. Since all terms of $\mathcal{F}^\mu(\varepsilon, s, v)$ are obviously L-periodic in x, and since $\mathcal{F}^\mu(0,0,0) = 0$, it suffices to prove that each term of $\mathcal{F}^\mu(\varepsilon, s, v)$ tends to zero in the norm $||\cdot||_{\mathcal{Z}}$ as $|\varepsilon| + |s| + ||v||_{\mathcal{Y}_\mu} \to 0$. Firstly, Lemma 17 and the fact that θ satisfies (66) imply

$$\exists C > 0,\, \forall \mu \in (0,1),\, \forall v \in \mathcal{Y}_\mu, \qquad ||w||_{\mathcal{Z}} \le \mu^{-1} C ||v||_{\mathcal{Y}_\mu} \xrightarrow[v \to 0]{} 0,$$

for any $w \in \{D^\alpha v, y^2 v, y\theta v, \theta^2 v\}$ and $|\alpha| \le 2$. Next, let us recall that v satisfies (78)—(79), and from Lemma (18), n^ε satisfies (89)—(90). As a result, since $|U(z)| \le 1$ the functions $U(z) \int_{\mathbb{R}} n^\varepsilon(x, y')dy'$ and $\int_{\mathbb{R}} v(z, x, y')dy'$ are uniformly bounded and satisfy (83) with $\rho = 1$ and $K_b = K_A$. From Lemma 17, we thus deduce

$$\left\| v(z,x,y) U(z) \int_{\mathbb{R}} n^\varepsilon(x,y')dy' \right\|_{\mathcal{Z}} \le \left(C_1 K_A + \frac{\pi}{2} \right) ||n^\varepsilon||_{Y^*} ||v||_{\mathcal{Y}_\mu} \xrightarrow[v \to 0]{} 0,$$

$$\left\| v(z,x,y) \int_{\mathbb{R}} v(z,x,y')dy' \right\|_{\mathcal{Z}} \le \left(C_1 K_A + \frac{\pi}{2} \right) ||v||^2_{\mathcal{Y}_\mu} \xrightarrow[v \to 0]{} 0. \tag{94}$$

We now look at the term $U(z) n^\varepsilon(x,y) \int_{\mathbb{R}} v(z,x,y')dy'$. Since $|U(z)| \le 1$, n^ε satisfies (88), and v satisfies (79), we have

$$U(z) n^\varepsilon(x,y) \int_{\mathbb{R}} v(z,x,y')dy' = \underbrace{n^\varepsilon(x,y) e^{-\kappa|z|}}_{\in \mathcal{Y}_\mu} \times \underbrace{U(z) \int_{\mathbb{R}} v(z,x,y') e^{\kappa|z|} dy'}_{\text{satisfies (83) with } \rho=1,\, K_b=K_A}.$$

Thus, thanks to (78) and (84), we have

$$\begin{aligned} \left\| U(z) n^\varepsilon(x,y) \int_{\mathbb{R}} v(z,x,y')dy' \right\|_{\mathcal{Z}} &\le \left(C_1 K_A + \frac{\pi}{2} \right) ||v||_{\mathcal{Y}_\mu} ||n^\varepsilon e^{-\kappa|z|}||_{\mathcal{Y}_\mu} \\ &\le K_\kappa \left(C_1 K_A + \frac{\pi}{2} \right) ||v||_{\mathcal{Y}_\mu} ||n^\varepsilon||_{Y^*} \xrightarrow[v \to 0]{} 0. \end{aligned}$$

Next, it is well-known that, since $\kappa < -\frac{1}{2}c_0 + \frac{1}{2}\sqrt{c_0^2 - 4\lambda_0}$,

$$\exists C_U > 0,\, \forall z \in \mathbb{R}, \quad U(1-U)(z), |U'(z)| \le C_U e^{-\kappa|z|}.$$

Therefore, from (86)—(88), there holds

$$||sU'(z) n^\varepsilon(x,y)||_{\mathcal{Z}} \le C_U\, |s|\, ||e^{-\kappa|z|} n^\varepsilon(x,y)||_{\mathcal{Z}} \le C_U\, |s|\, ||n^\varepsilon||_{Y^*} \xrightarrow[s \to 0]{} 0,$$

$$||U'(z) n^\varepsilon_x(x,y)||_{\mathcal{Z}} = ||U'(z)(n^\varepsilon - n^0)_x(x,y)||_{\mathcal{Z}} \le C_U K_\sigma ||n^\varepsilon - n^0||_{Y^*} \xrightarrow[\varepsilon \to 0]{} 0.$$

Finally, setting

$$b_\varepsilon(z,x) := U(z)(1-U(z)) e^{\kappa|z|} \left(\lambda_0 + \int_{\mathbb{R}} n^\varepsilon(x,y')dy' \right) = U(z)(1-U(z)) e^{\kappa|z|} \int_{\mathbb{R}} (n^\varepsilon(x,y') - n^0(y'))dy',$$

we have, since (89)—(90) holds, that $||b_\varepsilon||_{L^\infty(\mathbb{R}^2)} \le C_U \frac{\pi}{2} ||n^\varepsilon - n^0||_{Y^*}$ and satisfies (83) with $\rho = 2$ and $K_{b_\varepsilon} = K_A C_U ||n^\varepsilon - n^0||_{Y^*}$. Therefore from (84) we deduce

$$||b_\varepsilon(z,x) e^{-\kappa|z|} n^\varepsilon(x,y)||_{\mathcal{Y}_\mu} \le (C_2 K_{b_\varepsilon} + ||b_\varepsilon||_\infty) ||n^\varepsilon||_{Y^*} \underset{\varepsilon \to 0}{=} ||n^\varepsilon||_{Y^*} \times o(1).$$

Therefore $\mathcal{F}^\mu$ is well-defined and continuous at $(0,0,0)$.

We now compute $D_{(s,v)} \mathcal{F}^\mu(0,0,0)$, that is the Fréchet derivative of $\mathcal{F}^\mu$ along the (s,v) variables at point $(0,0,0)$. We have $\mathcal{F}^\mu(0,s,v) = \mathcal{L}^\mu(s,v) + \mathcal{R}(s,v)$ where $\mathcal{R}(s,v) = s v_z - v \int_{\mathbb{R}} v(z,x,y')dy'$ and

$$\begin{aligned} \mathcal{L}^\mu(s,v) &= v_{zz} + 2v_{xz} + (1+\mu) v_{xx} + v_{yy} + c_0 v_z + sU'(z) n^0(y) \\ &\quad + v\left(1 - A^2 y^2 + \lambda_0 U(z)\right) - U(z) n^0(y) \int_{\mathbb{R}} v(z,x,y')dy'. \end{aligned} \tag{95}$$

We readily check from (81) and (94) that $\mathcal{R}(s,v) = o\left(|s| + ||v||_{\mathcal{Y}_\mu}\right)$. The continuity of $\mathcal{L}^\mu$ is a consequence of the controls obtained above. Consequently, $D_{(s,v)} \mathcal{F}^\mu(0,0,0) = \mathcal{L}^\mu$. It remains to prove the continuity of $D_{(s,v)} \mathcal{F}^\mu$ around $(0,0,0)$. This results from similar arguments as above. Details are omitted. □

5.3 Bijectivity of $\mathcal{L}^\mu$

In this subsection we prove that, if $\mu > 0$ is small enough, $\mathcal{L}^\mu$ is bijective from $\mathbb{R} \times \mathcal{S}_\mu$ to $\mathcal{Z}$, where $\mathcal{S}_\mu$ is a subset of $\mathcal{Y}_\mu$ that will be determined later. We proceed by analysis and synthesis. Fix $f \in \mathcal{Z}$, and assume there exist $(s, v) \in \mathbb{R} \times \mathcal{Y}_\mu$ such that $\mathcal{L}^\mu(s, v) = f$. Naturally, s and v depend *a priori* on μ, but to ease the readability we shall omit this dependence in the notations.

5.3.1 Decoupling in x and y

Thanks to (74) and (76), we have $v(z, x, \cdot), f(z, x, \cdot) \in L^2(\mathbb{R})$ for all $z, x \in \mathbb{R}$. Since the family of eigenfunctions $(\Gamma_j)_{j\in\mathbb{N}}$ of Proposition 8 forms a Hilbert basis of $L^2(\mathbb{R})$, we can write

$$v(z,x,y) = \sum_{j=0}^{\infty} v_j(z,x)\Gamma_j(y), \qquad f(z,x,y) = \sum_{j=0}^{\infty} f_j(z,x)\Gamma_j(y), \tag{96}$$

where we used the notation (67) for v_j and f_j. Since $(v, f) \in \mathcal{Y}_\mu \times \mathcal{Z}$, all functions v_j, f_j are L-periodic in x, we may compute their Fourier coefficients in x:

$$v_j(z,x) = \sum_{n\in\mathbb{Z}} v_j^n(z)e_n(x), \qquad f_j(z,x) = \sum_{n\in\mathbb{Z}} f_j^n(z)e_n(x), \tag{97}$$

where we used the notation (68)—(69) for v_j^n, f_j^n, and e_n. Note that the equalities (96)—(97) correspond, *a priori*, to a convergence of the series in the $L^2(\mathbb{R})$ and $L^2(0, L)$ norms respectively. However, since $\gamma > 3 > 1$, we deduce from (75) and (77) that equalities in (97) hold pointwise. Additionally, $v_j \in C_b^2(\mathbb{R}^2)$ and $f_j \in C_b(\mathbb{R})$ with

$$||f_j||_{L^\infty(\mathbb{R}^2)} \le \frac{||f||_{\mathcal{Z}}}{(1+j)^\beta},$$

and the pointwise equality

$$D_z^p D_x^q v_j(z,x) = \sum_{n\in\mathbb{Z}} (v_j^n)^{(p)}(z)(i\sigma n)^q e_n(x), \qquad (p+q \le 2),$$

which leads to

$$||D_z^p D_x^q v_j||_{L^\infty(\mathbb{R}^2)} \le \frac{C||v||_{\mathcal{Y}_\mu}}{(1+j)^\beta}, \qquad (p+q\le 2),$$

for some $C = C(\gamma)$. Next, since $\beta > \frac{19}{4} > \frac{5}{4}$ and (25) holds, the series in (96) are also normally convergent, which leads to pointwise equalities in (96). Additionally, since $\beta > \frac{19}{4} > \frac{9}{4}$, with (26) we have the following pointwise equality:

$$D_z^p D_x^q D_y^r v(z,x,y) = \sum_{j\in\mathbb{N}} D_z^p D_x^q v_j(z,x)\Gamma_j^{(r)}(y), \qquad (p+q+r \le 2),$$

and since (24) holds, we also have

$$\int_{\mathbb{R}} v(z,x,y)dy = \sum_{j=0}^{\infty} v_j(z,x)\int_{\mathbb{R}}\Gamma_j(y)dy = \sum_{j=0}^{\infty} m_j v_j(z,x),$$

where we recall the notation $m_j := \int_{\mathbb{R}} \Gamma_j(y)dy$.

Let us recall that n^0 is given by (8) and that $\Gamma_j'' + (1 - A^2y^2)\Gamma_j = -\lambda_j\Gamma_j$ from Proposition 8. Consequently, when projecting the equation $\mathcal{L}^\mu(s, v) = f$ along Γ_j, we obtain

$$(v_j)_{zz} + 2(v_j)_{xz} + (1+\mu)(v_j)_{xx} + c_0(v_0)_z - (\lambda_j - \lambda_0 U(z))\, v_j = f_j, \qquad j \ge 1, \tag{98}$$

$$(v_0)_{zz} + 2(v_0)_{xz} + (1+\mu)(v_0)_{xx} + c_0(v_0)_z - \lambda_0\,(1 - 2U(z))\, v_0 = f_0 - \eta U'(z)s + \eta U(z)\sum_{\ell=1}^{\infty} m_\ell v_\ell. \tag{99}$$

Then, multiplying (98) and (99) by $\frac{1}{L}e_{-n}(x)$ and integrating over $x \in [0, L]$, we obtain

$$\mathcal{E}_{n,j,\mu}[v_j^n] := (v_j^n)'' + (2in\sigma + c_0)\,(v_j^n)' - \left(\lambda_j - (1+\delta_{0j})\lambda_0 U(z) + (1+\mu)n^2\sigma^2\right) v_j^n = \begin{cases} f_j^n(z) & j \geq 1, \\ \widetilde{f_0^n}(z) & j = 0, \end{cases} \tag{100}$$

where we recall $\sigma := \frac{2\pi}{L} > 0$ and denote

$$\widetilde{f_0^n}(z) := f_0^n(z) - \eta U'(z) s \delta_{n0} + \eta U(z) \sum_{\ell=1}^{\infty} m_\ell v_\ell^n(z). \tag{101}$$

Finally, we define the operator

$$\begin{aligned} \mathcal{L}_{n,j,\mu} \colon E_\kappa^2 &\to E_\kappa^0 \\ u &\mapsto \mathcal{E}_{n,j,\mu}[u], \end{aligned} \tag{102}$$

where for any $k \in \mathbb{N}$ we set

$$E_\kappa^k := \left\{ g \in C^k(\mathbb{R}, \mathbb{C}) : ||g||_{\kappa,k} < \infty \right\}, \qquad ||g||_{\kappa,k} := \sum_{r=0}^{k} ||g^{(r)}(z) e^{\kappa|z|}||_{L^\infty}. \tag{103}$$

The proof for the rest of subsection 5.3 is organized as follows.

- In subsection 5.3.2, we construct a fundamental system of solutions of the homogenous equations associated to (100).
- Then, in subsection 5.3.3, we fix the value of κ and investigate the injectivity of the linear operators $\mathcal{L}_{n,j,\mu}$. To ensure that each $\mathcal{L}_{n,j,\mu}$ is injective, we may redefine some of them on a smaller space $\mathcal{S}_{n,j,\mu} \subset E_\kappa^2$.
- Next, in subsection 5.3.4, for any $j \geq 1$, we construct explicitly the solution of (100), which proves the surjectivity of $\mathcal{L}_{n,j,\mu}$. We also prove that, for any $n \in \mathbb{Z}$, $j \geq 1$ and $\mu > 0$ small enough, v_j^n satisfies (128).
- Afterwards, in subsection 5.3.5, we prove that $\widetilde{f_0^n}$ satisfies a bound of the type (77). The construction of v_0^n then follows in the same way, except for the case $n = 0$, where we shall also prove the existence and uniqueness of s.
- Finally, in subsection 5.3.6, we prove the existence and uniqueness of $s \in \mathbb{R}$ and $v \in \mathcal{S}_\mu \subset \mathcal{Y}_\mu$ such that $\mathcal{L}^\mu(s, v) = f$, where $\mathcal{S}_\mu$ is constructed from the spaces $\mathcal{S}_{n,j,\mu}$.

5.3.2 Fundamental system of solutions for the homogeneous problem

We consider the homogeneous equation associated to (100), that is

$$k'' + (2in\sigma + c_0)\,k' - \left(\lambda_j - (1+\delta_{0j})\lambda_0 U(z) + (1+\mu)n^2\sigma^2\right) k = 0. \tag{104}$$

Although we assumed $0 < \mu < 1$, we also need to consider solutions of (104) for $\mu = 0$. For that reason we shall assume in this subsection that $0 \leq \mu < 1$ unless otherwise stated.

To construct a fundamental system of solutions of (104), we first take the limit $z \to \pm\infty$ in the coefficients of (104), and thus consider

$$k'' + (2in\sigma + c_0)\,k' - \left(\lambda_j - (1+\delta_{0j})\lambda_0 + (1+\mu)n^2\sigma^2\right) k = 0, \tag{105}$$

and

$$k'' + (2in\sigma + c_0)\,k' - \left(\lambda_j + (1+\mu)n^2\sigma^2\right) k = 0. \tag{106}$$

A fundamental system of solutions of (105) is given by $z \mapsto e^{a^{\pm}_{n,j,\mu} z}$ with[4]

$$a^{\pm}_{n,j,\mu} = \frac{1}{2}\left(-2in\sigma - c_0 \pm \sqrt{4\mu n^2\sigma^2 + c_0^2 + 4in\sigma c_0 + 4\left[(1-\delta_{0j})\lambda_j - \lambda_0\right]}\right). \tag{107}$$

Similarly, a system for (106) is given by $z \mapsto e^{b^{\pm}_{n,j,\mu} z}$ with

$$b^{\pm}_{n,j,\mu} = \frac{1}{2}\left(-2in\sigma - c_0 \pm \sqrt{4\mu n^2\sigma^2 + c_0^2 + 4in\sigma c_0 + 4\lambda_j}\right). \tag{108}$$

Note that, for all $(n, j) \in \mathbb{Z} \times \mathbb{N}$ and $0 \le \mu < 1$, one can straightforwardly check that

$$\mathrm{Re}\ a^-_{n,j,\mu} < 0 < \mathrm{Re}\ a^+_{n,j,\mu}, \qquad \mathrm{Re}\ b^-_{n,j,\mu} < 0, \tag{109}$$

$$\mathrm{sign}\left(\mathrm{Re}\ (b^+_{n,j,\mu})\right) = \mathrm{sign}\left(\lambda_j + (1+\mu)n^2\sigma^2\right), \tag{110}$$

$$\mathrm{Re}\left(a^+_{n,j,\mu} - a^-_{n,j,\mu}\right) > 0, \qquad \mathrm{Re}\left(b^+_{n,j,\mu} - b^-_{n,j,\mu}\right)\begin{cases} > 0 & \text{if } (n,j,c_0) \neq (0,0,c_0^*), \\ = 0 & \text{otherwise,}\end{cases} \tag{111}$$

with the convention $\mathrm{sign}(0) = 0$ and where we recall $c_0 \ge c_0^* := 2\sqrt{-\lambda_0}$. We have the following estimates.

Lemma 19 (Estimates related to $a^{\pm}_{n,j,\mu}, b^{\pm}_{n,j,\mu}$)**.** *There exist $\underline{C}, \overline{C} > 0$ such that for any $(n, j) \in \mathbb{Z} \times \mathbb{N}$ and $0 \le \mu < 1$, there holds*

$$|a^{\pm}_{n,j,\mu}|,\ |b^{\pm}_{n,j,\mu}| \le \overline{C}\left(1 + |n| + \sqrt{j}\right), \tag{112}$$

$$\left|a^+_{n,j,\mu} - a^-_{n,j,\mu}\right|,\ \left|b^+_{n,j,\mu} - b^-_{n,j,\mu}\right| \ge \underline{C}\sqrt{\mu n^2 + j + |n|}, \tag{113}$$

$$\left|\mathrm{Re}\, a^{\pm}_{n,j,\mu}\right|,\ \left|\mathrm{Re}\, b^{\pm}_{n,j,\mu}\right| \ge \underline{C}\sqrt{\mu n^2 + j + |n|} - c_0, \tag{114}$$

$$\mathrm{Re}\left(a^+_{n,j,\mu} - a^-_{n,j,\mu}\right),\ \mathrm{Re}\left(b^+_{n,j,\mu} - b^-_{n,j,\mu}\right) \ge \underline{C}\sqrt{j + |n|}, \tag{115}$$

$$|a^{\pm}_{n,j,\mu} - b^{\pm}_{n,j,\mu}| \le \overline{C}. \tag{116}$$

Proof. The proofs of estimates (112)—(115) are straightforward and omitted. As for (116), notice that

$$-2\left(a^-_{n,j,\mu} - b^-_{n,j,\mu}\right) = 2(a^+_{n,j,\mu} - b^+_{n,j,\mu}) = \sqrt{Z(n,j,\mu) - 4\lambda_0(1+\delta_{0j})} - \sqrt{Z(n,j,\mu)},$$

where $Z(n, j, \mu)$ belongs to the half-plane $H_+ := \{z \in \mathbb{C} : \mathrm{Re}\, z \ge 0\}$ for all n, j, μ. Therefore, setting $\Lambda := -4\lambda_0(1 + \delta_{0j}) \in \{-4\lambda_0, -8\lambda_0\} > 0$, it is enough to prove that the function $h\colon Z \in H_+ \mapsto \sqrt{Z + \Lambda} - \sqrt{Z}$ is uniformly bounded, which is rather clear. $\square$

The construction of solutions for (104) follows from the following when $(n, j) \neq (0, 0)$.

Lemma 20 (Fundamental system of (104) for $(n, j) \neq (0, 0)$)**.** *There exists $\mu_{max} > 0$ such that the following results hold. Fix any $(n, j) \in \mathbb{Z} \times \mathbb{N}$ with $(n, j) \neq (0, 0)$ and $0 \le \mu < \mu_{max}$. There exists a fundamental system of solutions (φ_-, φ_+) of (104) such that*

$$\varphi_-(z) = \begin{cases} P_-(z)e^{a^-_{n,j,\mu} z} & z \le 0, \\ Q_-(z)e^{b^-_{n,j,\mu} z} & z \ge 0,\end{cases} \qquad \varphi_+(z) = \begin{cases} P_+(z)e^{a^+_{n,j,\mu} z} & z \le 0, \\ Q_+(z)e^{b^+_{n,j,\mu} z} & z \ge 0,\end{cases} \tag{117}$$

with $P_\pm \in C^2_b(\mathbb{R}_-)$, $Q_\pm \in C^2_b(\mathbb{R}_+)$ and $a^{\pm}_{n,j,\mu}, b^{\pm}_{n,j,\mu}$ given by (107)—(108). Also, $\liminf_{z\to-\infty} |P_-(z)| > 0$ *and* $\liminf_{z\to+\infty} |Q_+(z)| > 0$.

[4]In what follows, for any $z = re^{i\theta} \in \mathbb{C}$ with $r \ge 0$ and $\theta \in]-\pi, \pi]$, we denote $\sqrt{z} := \sqrt{r}e^{i\theta/2}$. In particular, $\mathrm{Re}\ \sqrt{z} > 0$ if $z \in \mathbb{C}\backslash\mathbb{R}_-$.

Additionally, there exists $R_{max} > 0$ such that

$$\sup_{(n,j)\neq(0,0)} \sup_{0\leq\mu<\mu_{max}} \sup_{R\in\{P_\pm,Q_\pm\}} \left(||R||_\infty + ||R'||_\infty\right) \leq R_{max}, \tag{118}$$

where by convention the sup norm is taken over the domain of definition of R.

Next, there exists $W_0 > 0$ such that for all $(n,j) \neq (0,0)$ and $0 \leq \mu < \mu_{max}$, the Wronskian of (φ_-, φ_+) at $z = 0$ satisfies

$$|W_\varphi| := \left|\left[\varphi'_-\varphi_+ - \varphi'_+\varphi_-\right](0)\right| \geq W_0, \tag{119}$$

Also, there exist $C_W, N_0, J_0 > 0$ such that, if $|n| \geq N_0$ or $j \geq J_0$, we have for any such n, j

$$\frac{1}{|W_\varphi|} \leq \frac{C_W}{\sqrt{1+\mu n^2+j+|n|}}, \qquad \forall\mu\in[0,\mu_{max}). \tag{120}$$

Finally, there exist $\zeta_1, \zeta_2 > 0$ such that for all $(n,j) \neq 0$ and $0 \leq \mu < \mu_{max}$,

$$\int_{-\infty}^{0} |\varphi_+(z)|^2 dz \geq \zeta_1 e^{-\zeta_2 \operatorname{Re} a^+_{n,j,\mu}}. \tag{121}$$

The proof of Lemma 20, lengthy and technical, is postponed to Appendix A.1. The case $n = j = 0$ is simpler and reads as follows.

Lemma 21 (Fundamental system of (104) for $(n,j) = (0,0)$)**.** *For all $0 \leq \mu < 1$, a fundamental system of solutions of (104) when $n = j = 0$ is given by (U', Υ), where U solves (16) and*

$$\Upsilon(z) := U'(z) \int_z^{+\infty} \frac{1}{U'(\omega)^2} e^{-c_0\omega} d\omega. \tag{122}$$

Additionally, we have

$$U'(z) \approx_{-\infty} e^{a^+_{0,0,0}z}, \qquad U'(z) \approx_{+\infty} \begin{cases} e^{b^+_{0,0,0}z} & \text{if } c_0 > c_0^*, \\ z e^{b^+_{0,0,0}z} & \text{if } c_0 = c_0^*, \end{cases} \tag{123}$$

$$\Upsilon(z) \approx_{-\infty} e^{a^-_{0,0,0}z} \qquad \Upsilon(z) \approx_{+\infty} \begin{cases} e^{b^-_{0,0,0}z} & \text{if } c_0 > c_0^*, \\ \frac{1}{z} e^{b^-_{0,0,0}z} & \text{if } c_0 = c_0^*, \end{cases} \tag{124}$$

where $A(z) \approx_{\pm\infty} B(z)$ with $B(z) > 0$ means $0 < \liminf_{\pm\infty} \frac{|A(z)|}{B(z)} < \limsup_{\pm\infty} \frac{|A(z)|}{B(z)} < +\infty$.

Proof. Estimates (123) are classical results for the critical ($c_0 = c_0^*$) and supercritical ($c_0 > c_0^*$) Fisher-KPP traveling waves. When $n = j = 0$, (104) amounts to

$$k'' + c_0 k' - \lambda_0 \left(1 - 2U(z)\right) k = 0. \tag{125}$$

Note that μ does not play any role here. First, we see that U' solves (125) since U solves (16). In this case another solution (non-proportional to U') of (125) can be sought in the form of $\Upsilon(z) = g(z)U'(z)$. Using this, some straightforward computations yield that (122) is another solution. Then (124) follows straightforwardly from (122) and (123). □

5.3.3 Fixing the values κ and μ_{max}, redefinitions of $\mathcal{L}_{n,j,\mu}$

Here, we shall fix the value of κ with the following Lemma.

Lemma 22 (Choice of κ)**.** *If $\mu_{max} > 0$ is small enough, there exists $\kappa > 0$ such that for all $(n,j) \in \mathbb{Z}\times\mathbb{N}$ and $0 \leq \mu < \mu_{max}$, we have*

$$\begin{cases} \left|\operatorname{Re} a^\pm_{n,j,\mu}\right|, \left|\operatorname{Re} b^-_{n,j,\mu}\right| \geq 2\kappa, \\ \left|\operatorname{Re} b^+_{n,j,\mu}\right| \geq 2\kappa, & \text{if } \operatorname{Re} b^+_{n,j,\mu} < 0. \end{cases} \tag{126}$$

Also, there exist $C_\kappa, \overline{N}, \overline{J} \geq 0$ such that if $|n| \geq \overline{N}$ or $j \geq \overline{J}$, then for all $0 \leq \mu < \mu_{max}$, we have

$$0 < \frac{1}{|\operatorname{Re} a^\pm_{n,j,\mu}| - \kappa}, \frac{1}{|\operatorname{Re} b^\pm_{n,j,\mu}| - \kappa} \leq \frac{C_\kappa}{\sqrt{1+\mu n^2+j+|n|}}. \tag{127}$$

Proof. Let us first prove (126). We define the following sets:

$$\mathcal{I} := \left\{(n,j) \in \mathbb{Z}\times\mathbb{N} : \lambda_j + n^2\sigma^2 < 0\right\} = \left\{(n,j) \in \mathbb{Z}\times\mathbb{N} : \operatorname{Re} b^+_{n,j,0} < 0\right\},$$
$$\mathcal{R}_\mu := \left(\cup_{(n,j)\in\mathbb{Z}\times\mathbb{N}} \left\{\left|\operatorname{Re} a^+_{n,j,\mu}\right|, \left|\operatorname{Re} a^-_{n,j,\mu}\right|, \left|\operatorname{Re} b^-_{n,j,\mu}\right|\right\}\right) \bigcup \left(\cup_{(n,j)\in\mathcal{I}} \left\{\left|\operatorname{Re} b^+_{n,j,\mu}\right|\right\}\right),$$

for all $0 \le \mu < \mu_{max}$. Because of (109) and the definition of $\mathcal{I}$, there holds $0 \notin \mathcal{R}_0$. Note that, due to (114), the sets $\mathcal{I}$ and $\mathcal{R}_0 \cap [0, a^+_{0,0,0}]$ are finite. Therefore we have

$$m := \min\left(\mathcal{R}_0 \cap [0, a^+_{0,0,0}]\right) = \min \mathcal{R}_0 > 0.$$

Now, because of (114), there exist $N, J \ge 0$ such that for all $0 \le \mu < \mu_{max}$

$$\mathcal{R}_\mu \cap [0, a^+_{0,0,\mu}] \subset \cup_{|n|\le N, j\le J} \left\{\left|\operatorname{Re} a^+_{n,j,\mu}\right|, \left|\operatorname{Re} a^-_{n,j,\mu}\right|, \left|\operatorname{Re} b^-_{n,j,\mu}\right|, \left|\operatorname{Re} b^+_{n,j,\mu}\right|\right\}.$$

From (107)—(108) we easily obtain that

$$\sup_{|n|\le N,\, j\le J} \left|\operatorname{Re} a^\pm_{n,j,\mu} - \operatorname{Re} a^\pm_{n,j,0}\right| \xrightarrow[\mu\to 0]{} 0, \qquad \sup_{|n|\le N,\, j\le J} \left|\operatorname{Re} b^\pm_{n,j,\mu} - \operatorname{Re} b^\pm_{n,j,0}\right| \xrightarrow[\mu\to 0]{} 0.$$

As a result, taking μ_{max} small enough, we have

$$\inf_{0\le\mu<\mu_{max}} \min\left(\mathcal{R}_\mu \cap [0, a^+_{0,0,\mu}]\right) = \inf_{0\le\mu<\mu_{max}} \min \mathcal{R}_\mu \ge \frac{m}{2} > 0.$$

Consequently, (126) holds with $\kappa = \frac{m}{4} > 0$.

Finally, it remains to prove (127). Let $\underline{C} > 0$ being given by Lemma 19. There exist $\overline{N}, \overline{J} \ge 0$ such that, if $|n| \ge \overline{N}$ or $j \ge \overline{J}$, we have for all $0 \le \mu < \mu_{max}$

$$\sqrt{\mu n^2 + j + |n|} \ge \frac{2}{\underline{C}}(c_0 + \kappa) + 1.$$

We then deduce that

$$\frac{1}{\underline{C}\sqrt{\mu n^2 + j + |n|} - c_0 - \kappa} \le \frac{C_\kappa}{1 + \sqrt{\mu n^2 + j + |n|}} \le \frac{C_\kappa}{\sqrt{1 + \mu n^2 + j + |n|}},$$

for $C_\kappa = 2/\underline{C} > 0$. This yields (127) thanks to (114). ☐

Let us recall that $\mathcal{L}_{n,j,\mu}$ and E^k_κ are defined by (102) and (103) respectively. We equip E^2_κ with the Hermitian inner product $\langle g_1, g_2\rangle := \int_{\mathbb{R}} g_1(z)\overline{g_2(z)}dz$.

Lemma 23 (Injectivity of $\mathcal{L}_{n,j,\mu}$ after redefinitions)**.** *Let $\mu_{max} > 0$ small enough so that both Lemmas 20 and 22 hold. Let $(n,j) \neq (0,0)$, $\mu \in (0, \mu_{max})$ and $\kappa > 0$ given by Lemma 22.*

If $\operatorname{Re} b^+_{n,j,\mu} \ge 0$, then $\mathcal{L}_{n,j,\mu}$ is injective.

If $\operatorname{Re} b^+_{n,j,\mu} < 0$, then we set $\mathcal{S}_{n,j,\mu} := \{\varphi_+\}^\perp \subset E^2_\kappa$, and we redefine $\mathcal{L}_{n,j,\mu} \colon \mathcal{S}_{n,j,\mu} \to E^0_\kappa$ as an injective operator.

Finally, we set $\mathcal{S}_{0,0} := \{U'\}^\perp \subset E^2_\kappa$, and we redefine $\mathcal{L}_{0,0,\mu} \colon \mathcal{S}_{0,0} \to E^0_\kappa$ as an injective operator.

Proof. Let n, j, μ satisfy the above conditions. Let us recall that from Lemma 20, the solutions of $\mathcal{E}_{n,j,\mu}[u] = 0$ are exactly $C_-\varphi_- + C_+\varphi_+$ with $C_\pm \in \mathbb{C}$. Note that $\varphi_- \notin E^2_\kappa$ since $|\varphi_-(-\infty)| = +\infty$ from (109) and (117).

If $\operatorname{Re} b^+_{n,j,\mu} \ge 0$, then from (117) we also have $\varphi_+ \notin C_0(\mathbb{R}, \mathbb{C})$, so that $\varphi_+ \notin E^2_\kappa$, which implies $\ker \mathcal{L}_{n,j,\mu}$ is trivial.

If $\operatorname{Re} b^+_{n,j,\mu} < 0$, then from (117) and (126), we have $\varphi_+ \in E^2_\kappa$. Therefore $\ker \mathcal{L}_{n,j,\mu} = \operatorname{span}(\varphi_+)$. Setting $\mathcal{S}_{n,j,\mu} := \{\varphi_+\}^\perp$, we have that $\mathcal{L}_{n,j,\mu} \colon \mathcal{S}_{n,j,\mu} \to E^0_\kappa$ is injective.

The last assertion for $\mathcal{L}_{0,0,\mu}$ is proved similarly, using (123)—(124) and (126). ☐

5.3.4 Solving (100) when $j \geq 1$

For the rest of this section, we fix $\mu_{max}, \kappa > 0$ small enough such that Lemmas 20 and 22 are valid. From Lemma 20, we are equipped with (φ_-, φ_+) given by (117), which is a fundamental system of solutions of (104). Let us mention that by construction $\kappa < a^+_{0,0,0} = -\frac{1}{2}c_0 + \frac{1}{2}\sqrt{c_0^2 - 4\lambda_0}$, which is consistent with our assumption at the beginning of subsection 5.1.

In this subsection, we prove that, for each $n \in \mathbb{Z}$, $j \geq 1$, and $0 < \mu < \mu_{max}$ there exists a unique v_j^n such that $\mathcal{L}_{n,j,\mu} v_j^n = f_j^n$. Additionally, we shall prove the existence of $K > 0$ independent of n, j, μ, f such that

$$\left|(v_j^n)^{(k)}(z)\right| \leq K||f||_{\mathcal{Z}} \frac{e^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1+|n|^k + j^{k/2}}{1+\mu n^2 + j + |n|}, \quad k \in \{0,1,2\}, \forall z \in \mathbb{R}. \tag{128}$$

Let us recall that f satisfies (77), and thus $f_j^n \in E_\kappa^0$. In what follows, we denote $a^\pm = a^\pm_{n,j,\mu}$ and $b^\pm = b^\pm_{n,j,\mu}$ when there is no confusion. We shall split the proof in two subcases, depending on the sign of $\operatorname{Re} b^+$.

Indexes (n, j, μ) such that $j \geq 1$ and $\operatorname{Re} b^+_{n,j,\mu} \geq 0$. For such n, j, μ, we have the injectivity of $\mathcal{L}_{n,j,\mu} \colon E_\kappa^2 \to E_\kappa^0$ from Lemma 23, so there is at most one solution $v_j^n \in E_\kappa^2$ of (100). To prove its existence, we construct explicitly a solution with the variation of the constant, that is

$$v_j^n(z) = \varphi_-(z) \int_{-\infty}^z \frac{1}{W(\omega)} \varphi_+(\omega) f_j^n(\omega) d\omega + \varphi_+(z) \int_z^{+\infty} \frac{1}{W(\omega)} \varphi_-(\omega) f_j^n(\omega) d\omega,$$

where we denote the Wronskian $W(\omega) := \left[\varphi_-' \varphi_+ - \varphi_+' \varphi_-\right](\omega) \neq 0$. Also, notice that since (φ_-, φ_+) solve (100), there holds

$$W(\omega) = W(0) e^{-(c_0 + 2in\sigma)\omega} = W(0) e^{(a^+ + a^-)\omega} = W(0) e^{(b^+ + b^-)\omega}.$$

To prove that v_j^n satisfies $\mathcal{L}_{n,j,\mu} v_j^n = f_j^n$, it suffices to prove that $v_j^n \in E_\kappa^2$. It is in particular enough to prove that v_j^n satisfies (128).

Let us first prove that (128) holds for $k = 0$. For all $z \geq 0$, there holds

$$\begin{aligned} v_j^n(z) = {} & Q_-(z) e^{b^- z} \left(\int_{-\infty}^0 \frac{P_+(\omega)}{W_\varphi} e^{-a^-\omega} f_j^n(\omega) + \int_0^z \frac{Q_+(\omega)}{W_\varphi} e^{-b^-\omega} f_j^n(\omega) d\omega \right) \\ & + Q_+(z) e^{b^+ z} \left(\int_z^{+\infty} \frac{Q_-(\omega)}{W_\varphi} e^{-b^+\omega} f_j^n(\omega) d\omega \right), \end{aligned}$$

where we recall that $W(0) = W_\varphi$ satisfies (119). Combining (77), (118) and (126), we obtain

$$\begin{aligned} |v_j^n(z)| \leq {} & \frac{R_{max}^2 ||f||_{\mathcal{Z}}}{|W_\varphi| (1+j)^\beta (1+|n|)^\gamma} \times \\ & \left[e^{\operatorname{Re} b^- z} \left(\int_{-\infty}^0 e^{-\operatorname{Re} a^- \omega} e^{\kappa\omega} d\omega + \int_0^z e^{-\operatorname{Re} b^- \omega} e^{-\kappa\omega} d\omega \right) + e^{\operatorname{Re} b^+ z} \int_z^{+\infty} e^{-\operatorname{Re} b^+ \omega} e^{-\kappa\omega} d\omega \right] \\ \leq {} & \frac{R_{max}^2 ||f||_{\mathcal{Z}}}{|W_\varphi| (1+j)^\beta (1+|n|)^\gamma} \times \left(\frac{e^{\operatorname{Re} b^- z}}{-\operatorname{Re} a^- + \kappa} + \frac{e^{-\kappa z} - e^{\operatorname{Re} b^- z}}{-\operatorname{Re} b^- - \kappa} + \frac{e^{-\kappa z}}{\operatorname{Re} b^+ + \kappa} \right) \\ \leq {} & \frac{R_{max}^2 ||f||_{\mathcal{Z}}}{|W_\varphi| (1+j)^\beta (1+|n|)^\gamma} \left(\frac{1}{-\operatorname{Re} a^- + \kappa} + \frac{1}{-\operatorname{Re} b^- - \kappa} + \frac{1}{\operatorname{Re} b^+ + \kappa} \right) e^{-\kappa z}. \end{aligned}$$

Let $\overline{N}_0 = \max(N_0, \overline{N})$ and $\overline{J}_0 = \max(J_0, \overline{J})$, where N_0, J_0 are given by Lemma 20 and $\overline{N}, \overline{J}$ are given by Lemma 22. If $|n| \geq \overline{N}_0$ or $j \geq \overline{J}_0$, then (120) and (127) hold. Therefore

$$|v_j^n(z)| \leq \frac{3 C_W C_\kappa R_{max}^2 ||f||_{\mathcal{Z}} e^{-\kappa z}}{(1+j)^\beta (1+|n|)^\gamma} \times \frac{1}{1 + \mu n^2 + j + |n|}, \qquad \text{if } |n| \geq \overline{N}_0 \text{ or } j \geq \overline{J}_0. \tag{129}$$

Meanwhile, if $|n| \leq \overline{N}_0$ and $j \leq \overline{J}_0$, we have from (119) and (126) that

$$|v_j^n(z)| \leq \frac{3R_{max}^2||f||_{\mathcal{Z}}e^{-\kappa z}}{\kappa W_0(1+j)^\beta(1+|n|)^\gamma}, \qquad \text{if } |n| \leq \overline{N}_0 \text{ and } j \leq \overline{J}_0. \tag{130}$$

Note that $\overline{N}_0, \overline{J}_0$ do not depend on $\mu \in (0, \mu_{max})$. Therefore, combining (129)—(130), there exists $K > 0$ independent of n, j, μ, f such that (128) holds for $k = 0$, $z \geq 0$. The proof is similar for $z \leq 0$.

Let us now prove that (128) is valid for $k = 1$. Note that

$$(v_j^n)'(z) = \varphi_-'(z) \int_{-\infty}^z \frac{1}{W(\omega)} \varphi_+(\omega) f_j^n(\omega) d\omega + \varphi_+'(z) \int_z^{+\infty} \frac{1}{W(\omega)} \varphi_-(\omega) f_j^n(\omega) d\omega.$$

Then similar calculations and arguments yield that if $|n| \geq \overline{N}_0$ or $j \geq \overline{J}_0$, then for any $z \in \mathbb{R}$

$$\left|(v_j^n)'(z)\right| \leq \frac{3C_W C_\kappa R_{max}^2||f||_{\mathcal{Z}}e^{-\kappa z}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1 + \max\left(|a^+|, |a^-|, |b^+|, |b^-|\right)}{\left(1 + \sqrt{\mu n^2 + j + |n|}\right)^2},$$

thus, using (112), we obtain

$$|(v_j^n)'(z)| \leq \frac{3C_W C_\kappa (1+\overline{C}) R_{max}^2||f||_{\mathcal{Z}}e^{-\kappa z}}{(1+j)^\beta(1+|n|)^\gamma} \times \frac{1 + |n| + \sqrt{j}}{1 + \mu n^2 + j + |n|}.$$

Meanwhile, if $|n| \leq \overline{N}_0$ and $j \leq \overline{J}_0$, then in the same fashion, for any $z \in \mathbb{R}$, there holds

$$\begin{aligned} |(v_j^n)'(z)| &\leq \frac{3R_{max}^2||f||_{\mathcal{Z}}e^{-\kappa z}}{\kappa W_0(1+j)^\beta(1+|n|)^\gamma} \times \left[1 + \max\left(|a^+|, |a^-|, |b^+|, |b^-|\right)\right] \\ &\leq \frac{3R_{max}^2||f||_{\mathcal{Z}}e^{-\kappa z}}{\kappa W_0(1+j)^\beta(1+|n|)^\gamma} \left(1 + \max_{|n|\leq\overline{N}_0, j\leq\overline{J}_0, 0\leq\mu\leq\mu_{max}} \max\left(|a_{n,j,\mu}^+|, |a_{n,j,\mu}^-|, |b_{n,j,\mu}^+|, |b_{n,j,\mu}^-|\right)\right). \end{aligned}$$

Likewise, taking $K > 0$ possibly even larger, v_j^n satisfies (128) for $k = 1$. Finally, since (128) is proved for $k \in \{0, 1\}$, the proof for $k = 2$ is a direct consequence, with a possibly larger $K > 0$, since v_j^n solves (100) and (77) holds. Therefore, assuming $j \geq 1$ and $\operatorname{Re} b_{n,j,\mu}^+ \geq 0$, (128) holds with $K > 0$ that does not depend on n, j, μ, f.

Indexes (n, j, μ) such that $j \geq 1$ and $\operatorname{Re} b_{n,j,\mu}^+ < 0$. For such n, j, μ, from Lemma 23, the operator $\mathcal{L}_{n,j,\mu} \colon \mathcal{S}_{n,j,\mu} \to E_\kappa^0$ is injective with $\mathcal{S}_{n,j,\mu} = \{\varphi_+\}^\perp$. We define the family

$$\chi_\xi(z) := \xi\varphi_+(z) + \varphi_-(z) \int_{-\infty}^z \frac{1}{W(\omega)} \varphi_+(\omega) f_j^n(\omega) d\omega + \varphi_+(z) \int_0^z \frac{1}{W(\omega)} \varphi_-(\omega) f_j^n(\omega) d\omega, \qquad \xi \in \mathbb{C}.$$

Owing to the variation of the constant, we see that χ_ξ solves (100). Using (77), (118)—(119) and (126) as we did above, one can readily check that $\chi_\xi \in E_\kappa^2$ for all $\xi \in \mathbb{C}$. Thus there is a unique $\xi_0 = -\frac{\langle \chi_0, \varphi_+ \rangle}{\langle \varphi_+, \varphi_+ \rangle} \in \mathbb{C}$ such that $\chi_{\xi_0} \in \mathcal{S}_{n,j,\mu}$. Therefore, the equation $\mathcal{L}_{n,j,\mu} v_j^n = f_j^n$ admits a unique solution, given by $v_j^n = \chi_{\xi_0}$.

It remains to prove that v_j^n satisfies (128). First, notice that $\operatorname{Re} b_{n,j,\mu}^+ < 0$ implies, from (110), that (n, j) belongs to a finite set $S \subset \mathbb{Z} \times \mathbb{N}_+$, independently of $\mu \in (0, \mu_{max})$. Fix now $(n, j) \in S$. It can be readily checked that there exists $C_{n,j} > 0$ independent of f such that

$$||\varphi_+||_{\kappa,2} \leq C_{n,j}, \quad ||\chi_0||_{\kappa,2} \leq C_{n,j}||f||_{\mathcal{Z}}, \quad \forall \mu \in (0, \mu_{max}).$$

We claim that there exists $C'_{n,j} > 0$ independent of f such that $|\xi_0| \leq C'_{n,j}||f||_{\mathcal{Z}}$ for all $\mu \in (0, \mu_{max})$. On the one hand, by the Cauchy-Schwarz inequality

$$|\langle \chi_0, \varphi_+ \rangle| \leq \sqrt{\int_{\mathbb{R}} |\chi_0(z)|^2 dz \int_{\mathbb{R}} |\varphi_+(z)|^2 dz} \leq \frac{1}{\kappa}||\chi_0||_{\kappa,0}||\varphi_+||_{\kappa,0} \leq \frac{1}{\kappa} C_{n,j}^2 ||f||_{\mathcal{Z}}.$$

On the other hand, we have from (121)

$$|\langle\varphi_+,\varphi_+\rangle| \geq \zeta_1 e^{-\zeta_2 \operatorname{Re} a^+} \geq \zeta_1 e^{-\zeta_2 M}, \qquad M := \max_{(n,j)\in S} \max_{0\leq\mu\leq 1} \operatorname{Re} a^+_{n,j,\mu} > 0.$$

Therefore we deduce that such $C'_{n,j}$ exists. Thus, we have $||v_j^n||_{\kappa,2} \leq \left(1 + C'_{n,j}\right) C_{n,j}||f||_{\mathcal{Z}}$ for all $(n,j) \in S$ and $0 < \mu < \mu_{max}$. Since the set S is finite, taking $K > 0$ possibly even larger, independently of n, j, μ, f, we deduce that v_j^n satisfies (128).

5.3.5 Solving (100) when $j = 0$

From subsection 5.3.4, we are now equipped with $v_j^n = \mathcal{L}^{-1}_{n,j,\mu}(f_j^n)$ for every $n \in \mathbb{Z}$, $j \geq 1$ and $0 < \mu < \mu_{max}$. Also, there exists $K > 0$ independent of n, j, μ, f such that those v_j^n satisfy (128). Therefore, since (24) holds and $\beta > \frac{19}{4} > \frac{5}{4}$, we have

$$\left|\sum_{\ell=1}^{\infty} m_\ell v_\ell^n(z)\right| \leq K||f||_{\mathcal{Z}} \frac{e^{-\kappa|z|}}{(1+|n|)^\gamma} \times \Sigma, \qquad \Sigma := \sum_{\ell=1}^{\infty} \frac{|m_\ell|}{(1+j)^\beta} < \infty.$$

Let us recall that $\widetilde{f_0^n}$ is defined by (101). Since (77) holds, we deduce that

$$\exists C > 0,\, \forall n \neq 0,\, \forall \mu \in (0, \mu_{max}) \quad \left|\widetilde{f_0^n}(z)\right| \leq C||f||_{\mathcal{Z}} \frac{e^{-\kappa|z|}}{(1+|n|)^\gamma}, \tag{131}$$

As a consequence, for any $n \neq 0$, we prove, in the same manner as in subsection 5.3.4, that v_0^n satisfies (128) if $K > 0$ is large enough, independently of n, μ, f.

The case $n = j = 0$ is particular since this is the only equation where s, i.e. our perturbed speed, appears. Given that $\mathcal{L}_{0,0,\mu}$ does not depend on μ, we denote it $\mathcal{L}_{0,0}$ from now on. Let us recall that from Lemma 23, $\mathcal{L}_{0,0} \colon \mathcal{S}_{0,0} \to E^0_\kappa$ is injective. Repeating the same arguments as above, we prove that $\mathcal{L}_{0,0}$ is surjective, thus bijective. Now, we set

$$h(z) := \mathcal{L}^{-1}_{0,0}(U'), \qquad \widehat{\mathcal{S}}_{0,0} := \left\{U'\right\}^\perp \cap \{h\}^\perp \subset E^2_\kappa, \tag{132}$$

and we define the following operator as a restriction of $\mathcal{L}_{0,0}$:

$$\widehat{\mathcal{L}}_{0,0} \colon \widehat{\mathcal{S}}_{0,0} \subset E^2_\kappa \to E^0_\kappa.$$

It is clear that $\widehat{\mathcal{L}}_{0,0}$ is not bijective since $\widehat{\mathcal{S}}_{0,0} \subsetneq \mathcal{S}_{0,0}$. However, we shall prove that the linear operator

$$\begin{aligned} \mathcal{M} \colon \mathbb{C} \times \widehat{\mathcal{S}}_{0,0} &\to E^0_\kappa \\ (s,v) &\mapsto \eta U'(z)s + \widehat{\mathcal{L}}_{0,0} v \end{aligned}$$

is bijective. Assume that $\mathcal{M}(s,v) = 0$. Then

$$\widehat{\mathcal{L}}_{0,0} v = \mathcal{L}_{0,0} v = -\eta U'(z)s,$$

which implies $v = -\eta s h$. Since $\widehat{\mathcal{S}}_{0,0} \subset \{h\}^\perp$, we deduce that $s = 0$, thus $v = 0$. Therefore $\mathcal{M}$ is injective. Let us now prove that $\mathcal{M}$ is surjective. For any $f \in E^0_\kappa$, we set

$$s = \frac{\langle \mathcal{L}^{-1}_{0,0} f, h\rangle}{\eta\langle h, h\rangle}, \qquad v = \mathcal{L}^{-1}_{0,0}(f - \eta s U') = \mathcal{L}^{-1}_{0,0} f - \eta s h.$$

By definition of $\mathcal{L}_{0,0}$, we have $v \in \{U'\}^\perp$, thus $v \in \widehat{\mathcal{S}}_{0,0}$ by our choice of s. Finally, $\widehat{\mathcal{L}}_{0,0} v = \mathcal{L}_{0,0} v = f - \eta s U'$, so that we indeed have $\mathcal{M}(s,v) = f$. Hence $\mathcal{M}$ is bijective.

To conclude, we return to (100) for $n = j = 0$. Note that, with $f \in \mathcal{Z}$ being given, the functions $(v_\ell^0)_{\ell \geq 1}$ are uniquely determined from subsection 5.3.4. From now on, we rewrite $\mathcal{L}_{0,\ell} := \mathcal{L}_{0,\ell,\mu}$ since those operators do not, in fact, depend on μ. Therefore we may recast (100) as

$$\eta U'(z)s + \mathcal{E}_{0,0,0}[v_0^0] = f_0^0(z) + \eta U(z) \sum_{\ell=1}^{\infty} m_\ell \mathcal{L}_{0,\ell}^{-1}(f_\ell^0) =: \Phi_f(z). \tag{133}$$

From the bijectivity of $\mathcal{M}$, there thus exists a unique couple $(s, v_0^0) \in \mathbb{C} \times \widehat{\mathcal{S}}_{0,0}$ solving (133). We claim that

$$s \in \mathbb{R}, \qquad |s| \leq K_0 ||f||_{\mathcal{Z}}, \qquad |v_0^0(z)| \leq K_0 ||f||_{\mathcal{Z}} e^{-\kappa|z|}, \quad \forall z \in \mathbb{R}, \tag{134}$$

for some $K_0 > 0$ independent of f, μ. On the one hand, from (67)—(69), we see that f_ℓ^0 is real-valued for all $\ell \in \mathbb{N}$. On the other hand, note that for any $\ell \in \mathbb{N}$, $\mathcal{L}_{0,\ell}$ has real coefficients. By uniqueness of the solution of $\mathcal{L}_{0,\ell} v_\ell^0 = f_\ell^0$ for all $\ell \geq 1$, the functions v_ℓ^0 are also real-valued. Therefore $\Phi_f(z) \in \mathbb{R}$ and does not depend on μ. Also, repeating the same arguments that we used to obtain (131), we have $||\Phi_f||_{\kappa,0} \leq C_\Phi ||f||_{\mathcal{Z}}$ for some $C_\Phi > 0$ independent of f, μ. Now, inverting $\mathcal{M}$, we obtain

$$s = \frac{\int_{\mathbb{R}} \left[\mathcal{L}_{0,0}^{-1} \Phi_f\right](z) h(z) dz}{\eta \int_{\mathbb{R}} |h(z)|^2 dz}. \tag{135}$$

Similarly as above, $h = \mathcal{L}_{0,0}^{-1}(U')$ is real-valued and does not depend on μ. Therefore $s \in \mathbb{R}$ does not depend on μ, and there holds

$$|s| \leq \frac{C_\Phi ||\mathcal{L}_{0,0}^{-1}|| \, ||h||_{\kappa,0}}{\kappa\eta \int_{\mathbb{R}} |h(z)|^2 dz} ||f||_{\mathcal{Z}}. \tag{136}$$

Therefore, s satisfies (134) for K_0 large enough. One can readily check that the same is true for $v_0^0 = \mathcal{L}_{0,0}^{-1}(\Phi_f) - \eta s h$.

Combining the results of subsections 5.3.4 and 5.3.5, we have thus proved the following.

Proposition 24 (Results of subsections 5.3.4 and 5.3.5)**.** *There exist $\mu_{max}, \kappa > 0$ so that for any fixed $\mu \in (0, \mu_{max})$, the following results hold: there exists a finite set $I_\mu \subset \mathbb{Z} \times \mathbb{N}$, and a family of subsets $(\mathcal{S}_{n,j,\mu})_{(n,j) \in I_\mu}$ of E_κ^2, such that there exist a unique $s \in \mathbb{R}$ and, for any $(n,j) \in \mathbb{Z} \times \mathbb{N}$, a unique*

$$v_j^n \begin{cases} \in \mathcal{S}_{n,j,\mu} & \text{if } (n,j) \in I_\mu, \\ \in E_\kappa^2 & \text{otherwise,} \end{cases}$$

such that (100) holds. Additionally, there exists $K > 0$ independent of n, j, μ, f such that (128) holds.

Finally, with h, Φ_f being defined by (132)—(133), the real s is given by (135), satisfies (136), and does not depend on μ.

5.3.6 Reconstruction of $v = v_\mu$ so that $\mathcal{L}^\mu(s, v_\mu) = f$

The set $\mathcal{S}_\mu$. Let us fix $0 < \mu < \mu_{max}$. Let us recall that e_n is defined by (69). We set

$$\mathcal{S}_\mu := \bigcap_{(n,j) \in I_\mu} \left\{(z,x,y) \mapsto V(z) e_n(x) \Gamma_j(y) : V \in \mathcal{S}_{n,j,\mu}^\perp\right\}^\perp \subset \mathcal{Y}_\mu,$$

the second orthogonal being taken according to the following hermitian product on $\mathcal{Y}_\mu$:

$$\langle u, v \rangle_{\mathcal{Y}_\mu} = \int_{\mathbb{R}} \int_0^L \int_{\mathbb{R}} u(z,x,y) \overline{v}(z,x,y) dy dx dz.$$

Since I_μ is finite, it is clear that $\mathcal{S}_\mu$ is non-empty. Furthermore, $\mathcal{S}_\mu$ is closed for the topology associated to $\langle \cdot, \cdot \rangle$, and also for the topology of $\mathcal{Y}_\mu$, by virtue of the dominated convergence theorem. Therefore $\mathcal{S}_\mu$ is a Banach space when equipped with $|| \cdot ||_{\mathcal{Y}_\mu}$ defined by (72). Note also that $\mathcal{F}^\mu$ redefined as a function of $\mathbb{R} \times \mathbb{R} \times \mathcal{S}_\mu$ to $\mathcal{Z}$ still satisfies conditions (i)—(ii) of Theorem 7, since we only restrict the departure space.

Bijectivity of $\mathcal{L}^\mu$. Let us prove that $\mathcal{L}^\mu\colon \mathbb{R}\times\mathcal{S}_\mu \to \mathcal{Z}$ given by (95) is bijective. From Proposition 24 and (96)—(97), we already have that $\mathcal{L}^\mu$ is injective. Let us now prove that $\mathcal{L}^\mu$ is surjective. Since most of the arguments were already used in subsection 4.2, we only give a short proof. We are equipped with s and $v_j^n = v_j^n(z)$ provided by Proposition 24. Notice that (128) implies

$$\left|(v_j^n)^{(k)}(z)\right| \leq K||f||_{\mathcal{Z}} \frac{1}{(1+j)^\beta(1+|n|)^{\gamma+1-k}}, \qquad \forall\mu\in(0,\mu_{max}),\ \forall k\leq 2. \tag{137}$$

Now, we define

$$v_\mu(z,x,y) := \sum_{j=0}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} v_j^n(z)e_n(x)\right)\Gamma_j(y). \tag{138}$$

Because (25)—(26) hold and v_j^n satisfies (137) with $\beta > \frac{19}{4} > \frac{9}{4}$ and $\gamma > 3 > 2$, the function v_μ is well-defined, L-periodic in x and belongs to $C^2(\mathbb{R}^3)$ with

$$D_z^p D_x^q D_y^r v_\mu(z,x,y) = \sum_{j=0}^{+\infty}\left(\sum_{n=-\infty}^{+\infty}(v_j^n)^{(p)}(z)(in\sigma)^q e_n(x)\right)\Gamma_j^{(r)}(y), \qquad p+q+r\leq 2.$$

Similarly, using (27), since $\beta > \frac{19}{4} > \frac{17}{4}$ and $\gamma > 3 > 2$, and because K does not depend on μ, f, there holds

$$\exists C>0,\ \forall\mu\in(0,\mu_{max}), \quad |D^\alpha v_\mu(z,x,y)| \leq C||f||_{\mathcal{Z}}\frac{e^{-\kappa|z|}}{(1+y^2)^2}, \tag{139}$$

for any $|\alpha|\leq 2$ and $(z,x,y)\in\mathbb{R}^3$. Thus $v_\mu\in\mathcal{Y}_\mu$. By construction $v_\mu\in\mathcal{S}_\mu$ and satisfies $\mathcal{L}^\mu(s,v_\mu)=f$. Therefore $\mathcal{L}^\mu$ is bijective.

Boundedness of $||(\mathcal{L}^\mu)^{-1}||$ w.r.t. μ. From Proposition 24 and (139), we see that

$$||v_\mu||_{\mathcal{Y}_\mu} \leq (C+K)||f||_{\mathcal{Z}}, \quad \forall\mu\in(0,\mu_{max}),$$

where $C, K>0$ do not depend on f. Meanwhile, s satisfies a similar estimate in (136) and does not depend on μ. As a consequence,

$$\exists C_{\mathcal{L}}>0,\ \forall\mu\in(0,\mu_{max}), \quad ||(\mathcal{L}^\mu)^{-1}|| \leq C_{\mathcal{L}}. \tag{140}$$

5.4 Construction of $(s_{\varepsilon,\mu}, v_{\varepsilon,\mu})$ solving $\mathcal{F}^\mu(\varepsilon,s,v)=0$

Let us fix $\mu\in(0,\mu_{max})$, and recall that $\varepsilon_0^*>0$ has been fixed by Lemma 18. From subsections 5.1, 5.2 and 5.3.6, we can apply Theorem 7 to the function $\mathcal{F}^\mu$ at the point $(0,0,0)$. Hence there are $0<\overline{\varepsilon}_0\leq\varepsilon_0^*$ and $r>0$ that depend *a priori* on μ, such that, for any $|\varepsilon|<\overline{\varepsilon}_0$, the following holds: there is a unique $s_{\varepsilon,\mu}\in\mathbb{R}$ and $v_{\varepsilon,\mu}\in\mathcal{S}_\mu\subset\mathcal{Y}_\mu$ for which $|s_{\varepsilon,\mu}|+||v_{\varepsilon,\mu}||_{\mathcal{Y}_\mu}\leq r$ and $\mathcal{F}^\mu(\varepsilon,s_{\varepsilon,\mu},v_{\varepsilon,\mu})=0$.

We shall now prove that $\overline{\varepsilon}_0, r$ can be selected independently of μ, which is crucial for letting $\mu\to 0$ in the next subsection. To do so we have to redo the proof of Theorem 7 in a more accurate way than depicted in [50], which warrants to be detailed here. Set

$$\begin{aligned} T_{\varepsilon,\mu}\colon \mathbb{R}\times\mathcal{S}_\mu &\to \mathbb{R}\times\mathcal{S}_\mu \\ (s,v) &\mapsto (s,v) - (\mathcal{L}^\mu)^{-1}\left(\mathcal{F}^\mu(\varepsilon,s,v)\right). \end{aligned}$$

It is clear that $\mathcal{F}^\mu(\varepsilon,s,v)=0$ if and only if (s,v) is a fixed point of $T_{\varepsilon,\mu}$. Now, notice that $\mathcal{F}^\mu(\varepsilon,s,v) = \mathcal{L}^\mu(s,v)+\mathcal{G}_\mu(\varepsilon,s,v)$, where

$$\begin{aligned} \mathcal{G}_\mu(\varepsilon,s,v) = sv_z + v&\left(2A^2\varepsilon y\theta(x) - A^2\varepsilon^2\theta(x)^2 - U(z)\int_{\mathbb{R}}(n^\varepsilon-n^0)(x,y')dy' - \int_{\mathbb{R}} v(z,x,y')dy'\right) \\ &- U(z)(n^\varepsilon-n^0)\int_{\mathbb{R}} v(z,x,y')dy' + sU'(z)(n^\varepsilon-n^0) + 2U'(z)n_x^\varepsilon \\ &+ U(z)(1-U(z))n^\varepsilon\int_{\mathbb{R}}(n^\varepsilon-n^0)(x,y')dy'. \end{aligned} \tag{141}$$

Therefore $T_{\varepsilon,\mu}(s,v) = -(\mathcal{L}^\mu)^{-1}\left(\mathcal{G}_\mu(\varepsilon,s,v)\right)$. Note that

$$\begin{aligned}\left[D_{(s,v)}\mathcal{G}_\mu(\varepsilon,s,v)\right](\tau,w) = {} & sw_z + \tau v_z + w\left(2A^2\varepsilon y\theta(x) - A^2\varepsilon^2\theta(x)^2 - U(z)\int_{\mathbb{R}}(n^\varepsilon - n^0)(x,y')dy'\right)\\ & - w\int_{\mathbb{R}} v(z,x,y')dy' - v\int_{\mathbb{R}} w(z,x,y')dy'\\ & - U(z)(n^\varepsilon - n^0)\int_{\mathbb{R}} w(z,x,y')dy' + \tau U'(z)(n^\varepsilon - n^0).\end{aligned}$$

Let us recall that θ satisfies (66) with $k+\delta > \gamma + \frac{1}{2}$. In particular, θ satisfies (83) with $\rho = k+\delta-\gamma > 1/2$ and $K_b = K_\theta$. Repeating the same arguments as in subsection 5.2, we have

$$||sw_z||_{\mathcal{Z}} \le C|s|\,||w||_{\mathcal{Y}_\mu}, \qquad ||\tau v_z||_{\mathcal{Z}} \le C|\tau|\,||v||_{\mathcal{Y}_\mu},$$
$$||y\theta w||_{\mathcal{Z}} \le \left(C'_\rho K_\theta + ||\theta||_\infty\right)||w||_{\mathcal{Y}_\mu}, \qquad ||\theta^2 w|| \le \left(C_\rho K_\theta + ||\theta||_\infty\right)||w||_{\mathcal{Y}_\mu},$$
$$\left\|Uw\int_{\mathbb{R}}(n^\varepsilon - n^0)(x,y')dy'\right\|_{\mathcal{Z}} \le \left(C_1K_A + \frac{\pi}{2}\right)||n^\varepsilon - n^0||_{Y^*}||w||_{\mathcal{Y}_\mu},$$
$$\left\|w\int_{\mathbb{R}} v(z,x,y')dy'\right\|_{\mathcal{Z}}, \left\|v\int_{\mathbb{R}} w(z,x,y')dy'\right\|_{\mathcal{Z}} \le \left(C_1K_A + \frac{\pi}{2}\right)||v||_{\mathcal{Y}_\mu}||w||_{\mathcal{Y}_\mu},$$
$$\left\|U(n^\varepsilon - n^0)\int_{\mathbb{R}} w(z,x,y')dy'\right\|_{\mathcal{Z}} \le K_\kappa\left(C_1K_A + \frac{\pi}{2}\right)||w||_{\mathcal{Y}_\mu}||n^\varepsilon - n^0||_{\mathcal{Y}_\mu},$$
$$\left\|\tau U'(n^\varepsilon - n^0)\right\|_{\mathcal{Z}} \le C_U|\tau|\,||n^\varepsilon - n^0||_{Y^*}.$$

Consequently, we have

$$\left\|D_{(s,v)}\mathcal{G}_\mu(\varepsilon,s,v)\right\| \le \mathrm{C}\left(||n^\varepsilon - n^0||_{Y^*} + |\varepsilon| + |\varepsilon|^2 + |s| + ||v||_{\mathcal{Y}_\mu}\right),$$

where, crucially, $\mathrm{C} > 0$ does not depend on μ. Fix now any $\ell > 0$. Since $||n^\varepsilon - n^0||_{Y^*} \xrightarrow[\varepsilon\to 0]{} 0$ from Lemma 18, we may select $0 < r < \ell$ small enough such that for all $\mu \in (0,\mu_{max})$, we have

$$\forall\mu \in (0,\mu_{max}), \quad \min\left(|\varepsilon|, ||(s,v)||_{\mathbb{R}\times\mathcal{S}_\mu}\right) \le r \Longrightarrow \left\|D_{(s,v)}\mathcal{G}_\mu(\varepsilon,s,v)\right\| \le \frac{1}{2C_{\mathcal{L}}}. \tag{142}$$

Then, using (87), we have

$$\begin{aligned}||\mathcal{G}_\mu(\varepsilon,0,0)||_{\mathcal{Z}} &= \left\|2U'n^\varepsilon_x + U(1-U)n^\varepsilon\int_{\mathbb{R}}(n^\varepsilon - n^0)(x,y')dy'\right\|_{\mathcal{Z}}\\ &\le 2C_UK_\sigma||n^\varepsilon - n^0||_{Y^*} + \left(C_1K_A + \frac{\pi}{2}\right)C_U||n^\varepsilon - n^0||_{Y^*},\end{aligned}$$

and thus we may select $0 < \overline{\varepsilon}_0 < \min(r,\varepsilon_0^*)$ small enough such that

$$\forall\mu \in (0,\mu_{max}), \quad \forall|\varepsilon| \le \overline{\varepsilon}_0, \qquad ||\mathcal{G}_\mu(\varepsilon,0,0)||_{\mathcal{Z}} \le \frac{1}{2C_{\mathcal{L}}}r. \tag{143}$$

Let $B_r = \left\{(s,v) \in \mathbb{R}\times\mathcal{S}_\mu : ||(s,v)||_{\mathbb{R}\times\mathcal{S}_\mu} \le r\right\}$ be a closed subset of the Banach space $\mathbb{R}\times\mathcal{S}_\mu$. Note that $\mathcal{G}_\mu, D_{(s,v)}\mathcal{G}_\mu$ are continuous at $(0,0)$. Then from Taylor's theorem, (140) and (142), we have for any $|\varepsilon| < \overline{\varepsilon}_0 < r$ and $(s,v),(s',v') \in B_r$,

$$\begin{aligned}\left\|T_{\varepsilon,\mu}(s,v) - T_{\varepsilon,\mu}(s',v')\right\|_{\mathbb{R}\times\mathcal{S}_\mu} &\le ||(\mathcal{L}^\mu)^{-1}||\,\left\|\mathcal{G}_\mu(\varepsilon,s,v) - \mathcal{G}_\mu(\varepsilon,s',v')\right\|_{\mathbb{R}\times\mathcal{S}_\mu}\\ &\le C_{\mathcal{L}}\sup_{0<\omega<1}\left\|D_{(s,v)}\mathcal{G}_\mu(\varepsilon,s+\omega(s'-s),v+\omega(v'-v))\right\|\\ &\qquad\times ||(s-s',v-v')||_{\mathbb{R}\times\mathcal{S}_\mu}\\ &\le \frac{1}{2}||(s-s',v-v')||_{\mathbb{R}\times\mathcal{S}_\mu}.\end{aligned}$$

Repeating this argument, along with (143), yields

$$\begin{aligned}\|T_{\varepsilon,\mu}(s,v)\|_{\mathbb{R}\times\mathcal{S}_\mu} &\le ||(\mathcal{L}^\mu)^{-1}||\ \|\mathcal{G}_\mu(\varepsilon,s,v)-\mathcal{G}_\mu(\varepsilon,0,0)\|_{\mathbb{R}\times\mathcal{S}_\mu}+||(\mathcal{L}^\mu)^{-1}||\ \|\mathcal{G}_\mu(\varepsilon,0,0)\|_{\mathbb{R}\times\mathcal{S}_\mu}\\ &\le \frac{1}{2}||(s,v)||_{\mathbb{R}\times\mathcal{S}_\mu}+\frac{1}{2}r\le r.\end{aligned}$$

Consequently $T_{\varepsilon,\mu}$ maps B_r into itself and is contractive, thus by the fixed-point theorem it admits a unique fixed point in B_r. In conclusion, for any $\ell>0$, we can select $0<\bar{\varepsilon}_0<r<\ell$ such that for each $|\varepsilon|<\bar{\varepsilon}_0$ and $\mu\in(0,\mu_{max})$, there exists a unique $(s_{\varepsilon,\mu},v_{\varepsilon,\mu})\in B_r$ satisfying $\mathcal{F}^\mu(\varepsilon,s_{\varepsilon,\mu},v_{\varepsilon,\mu})=0$. Since $\ell>0$ was taken arbitrarily, we also have

$$\sup_{0<\mu<\mu_{max}}||(s_{\varepsilon,\mu},v_{\varepsilon,\mu})||_{\mathbb{R}\times\mathcal{S}_\mu}\xrightarrow[\varepsilon\to 0]{}0. \tag{144}$$

5.5 Letting the parameter μ tend to zero

Note that so far we only used the fact that $\beta>\frac{19}{4}>\frac{17}{4}$ and $\gamma>3>2$ at most. Since by assumption $\beta>\frac{19}{4}$ and $\gamma>3$, we may redo the above proof by replacing $\mathcal{Y}_\mu,\mathcal{Z}$ in (70) and (71) with

$$\widehat{\mathcal{Y}}_\mu:=\left\{v\in C^3(\mathbb{R}^3)\ \middle|\ \begin{array}{c}v(z,x+L,y)=v(z,x,y),\quad\text{on }\mathbb{R}^3,\\ \exists C>0,\ \forall|\alpha|\le 3,\quad |D^\alpha v(z,x,y)|\le\frac{Ce^{-\kappa|z|}}{(1+y^2)^2}\quad\text{on }\mathbb{R}^3,\\ \exists K>0,\ \forall n\in\mathbb{Z},\ \forall j\in\mathbb{N},\ \forall z\in\mathbb{R},\quad\text{there holds}\\ |(v_j^n)^{(k)}(z)|\le\frac{Ke^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma}\times\frac{1+|n|^k+j^{k/2}}{1+\mu n^2+j+|n|},\quad k\le 3\end{array}\right\}, \tag{145}$$

and

$$\widehat{\mathcal{Z}}:=\left\{f\in C^1(\mathbb{R}^3)\ \middle|\ \begin{array}{c}f(z,x+L,y)=f(z,x,y),\quad\text{on }\mathbb{R}^3,\\ \exists C>0,\ \forall|\alpha|\le 1,\quad |D^\alpha f(z,x,y)|\le\frac{Ce^{-\kappa|z|}}{1+y^2}\quad\text{on }\mathbb{R}^3,\\ \exists K>0,\ \forall n\in\mathbb{Z},\ \forall j\in\mathbb{N},\ \forall z\in\mathbb{R},\quad\text{there holds}\\ |(f_j^n)^{(k)}(z)|\le\frac{Ke^{-\kappa|z|}}{(1+j)^\beta(1+|n|)^\gamma}\times\left(1+|n|^k+j^{k/2}\right),\quad k\le 1\end{array}\right\}, \tag{146}$$

equipped with the respective norms

$$\begin{aligned}||v||_{\widehat{\mathcal{Y}}_\mu}=&\sum_{|\alpha|\le 3}\left[\sup_{(z,x,y)\in\mathbb{R}^3}\left|(1+y^2)^2D^\alpha v(z,x,y)\right|e^{\kappa|z|}\right]\\ &+\sum_{k=0}^{3}\sup_{n\in\mathbb{Z},j\in\mathbb{N}}\left[(1+j)^\beta(1+|n|)^\gamma\frac{1+\mu n^2+j+|n|}{1+|n|^k+j^{k/2}}\sup_{z\in\mathbb{R}}\left|(w_j^n)^{(k)}(z)e^{\kappa|z|}\right|\right],\end{aligned}$$

$$\begin{aligned}||f||_{\widehat{\mathcal{Z}}}=&\sum_{|\alpha|\le 1}\left[\sup_{(z,x,y)\in\mathbb{R}^3}\left|(1+y^2)D^\alpha f(z,x,y)\right|e^{\kappa|z|}\right]\\ &+\sum_{k\in\{0,1\}}\sup_{n\in\mathbb{Z},j\in\mathbb{N}}\left[\frac{(1+j)^\beta(1+|n|)^\gamma}{1+|n|^k+j^{k/2}}\sup_{z\in\mathbb{R}}\left|(f_j^n)^{(k)}(z)e^{\kappa|z|}\right|\right].\end{aligned}$$

The proof in itself requires only slightly more precision, for example the Young inequality $\sqrt{j}|n|\le\frac{2}{3}j^{3/2}+\frac{1}{3}|n|^3$, or the proof of (84)—(85) which requires to split the summation over $m\in\mathbb{Z}$ into $m\le 0$, $m\ge n$ and $0\le m\le n$ (assuming $n\ge 0$). Details are omitted.

Let us fix $|\varepsilon|<\bar{\varepsilon}_0$. From subsections 5.1 to 5.4, for any $\mu\in(0,\mu_{max})$, we are thus equipped with $(s_{\varepsilon,\mu},v_{\varepsilon,\mu})\in\mathbb{R}\times\widehat{\mathcal{Y}}_\mu$, with $|s_{\varepsilon,\mu}|$, $||v_{\varepsilon,\mu}||_{\mathcal{Y}_\mu}\le r$ where r does not depend on μ. Therefore there exists a sequence $(\mu_m)_{m\in\mathbb{N}}$ in $(0,\mu_{max})$ that tends to zero such that $s_{\varepsilon,\mu_m} \xrightarrow[m\to+\infty]{} s_\varepsilon$ for some $s_\varepsilon \in [-r,r]$. On the other hand, if we define for any $k\in\mathbb{N}$,

$$C^k_{w,L}(\mathbb{R}^3) := \left\{ g \in C^k_b(\mathbb{R}^3) : g(z,x+L,y) = g(z,x,y) \text{ on } \mathbb{R}^3, \quad ||g||_{w,k} < \infty \right\},$$

$$w(z,y) := (1+y^2)^2 e^{\kappa|z|}, \quad ||g||_{w,k} := \sum_{|\alpha|=0}^{k} \sup_{(z,x,y)\in\mathbb{R}^3} |w(z,y) D^\alpha g(z,x,y)|,$$

we see that $||v_{\varepsilon,\mu_m}||_{w,3} \le ||v_{\varepsilon,\mu_m}||_{\widehat{\mathcal{Y}}_\mu} \le r$. We claim that a subsequence of $(v_{\varepsilon,\mu_m})_{m\in\mathbb{N}}$ converges, as $m\to+\infty$, to some $v_\varepsilon \in C^2_{w_0,L}(\mathbb{R}^3)$, with $w_0(z,x,y) := (1+y^2)e^{\frac{\kappa}{2}z}$. First, one can readily check that $C^2_{w_0,L}(\mathbb{R}^3)$ is complete for $||\cdot||_{w_0,2}$. Then, because $(v_{\varepsilon,\mu_m})_m$ is bounded in $C^3_{w,L}(\mathbb{R}^3)$, for any $\delta>0$ there exist $z_\delta, y_\delta \ge 0$ such that for all $m,n\in\mathbb{N}$ and $|\alpha|\le 2$,

$$|(D^\alpha v_{\varepsilon,\mu_m}(z,x,y) - D^\alpha v_{\varepsilon,\mu_n}(z,x,y))\, w_0(z,y)| \le \delta, \qquad \forall |z|\ge z_\delta,\ \forall |y|\ge y_\delta,\ \forall x\in[0,L].$$

From there, redoing the proof of the Arzelà-Ascoli theorem, we prove that with another extraction $(\mu'_m)_{m\in\mathbb{N}}$, we obtain for m,n large enough

$$\left|\left(D^\alpha v_{\varepsilon,\mu'_m}(z,x,y) - D^\alpha v_{\varepsilon,\mu'_n}(z,x,y)\right) w_0(z,y)\right| \le \delta, \qquad \forall (z,x,y)\in[-z_\delta,z_\delta]\times[0,L]\times[-y_\delta,y_\delta].$$

Consequently the sequence v_{ε,μ'_m} is uniformly Cauchy in $C^2_{w_0,L}(\mathbb{R}^3)$, thus convergent to some $v_\varepsilon \in C^2_{w_0,L}(\mathbb{R}^3)$.

Completion of the proof of Theorem 6. Let us fix $|\varepsilon|\le\overline{\varepsilon}_0$. By construction we have $\mathcal{F}^\mu(\varepsilon, s_{\varepsilon,\mu'_m}, v_{\varepsilon,\mu'_m}) = 0$ for all $m\in\mathbb{N}$. Passing to the limit as $m\to+\infty$, thanks to the dominated convergence theorem, we obtain $\mathcal{F}(\varepsilon,s_\varepsilon,v_\varepsilon)=0$. As a result,

$$u^\varepsilon(z,x,y) = U(z)n^\varepsilon(x,y) + v_\varepsilon(z,x,y), \qquad z = x-(c_0+s_\varepsilon)t,$$

solves (5) by construction, and satisfies (17). Finally, from (144) combined with

$$||v_\varepsilon||_{w_0,2} \le \limsup_{m\to+\infty} ||v_{\varepsilon,\mu'_m}||_{w_0,2} \le \sup_{m\in\mathbb{N}} ||v_{\varepsilon,\mu'_m}||_{\mathcal{Y}_\mu},$$

we deduce that $|s_\varepsilon|, ||v_\varepsilon||_{w_0,2}\to 0$ as $\varepsilon\to 0$. This yields (18) with $b=\frac{\kappa}{2}>0$. □

6 Insights of the results on the biological model

In this section, our goal is to discuss some biological implications of our mathematical analysis, completed by some numerical explorations, for a population facing a nonlinear environmental gradient.

Throughout this section, we assume $\theta\in C_b(\mathbb{R})$ and $0<A<1$ so that $\lambda_0<0$, meaning Theorems 3, 4 and 5 hold. Letting $\alpha := \sqrt{2A}$, (10), (11) and (12) are recast

$$n^\varepsilon(x,y) \approx n^0(y)\left(1+\varepsilon \underbrace{A\,\rho_\alpha * \theta(x)}_{\text{deformation}} y + \cdots\right), \quad \rho_\alpha(z) := \frac{1}{2}\alpha e^{-\alpha|z|}. \tag{147}$$

In the following, we discuss two types of error between $n^\varepsilon(x,y)$ and $n^0(y)$: firstly, the so-called relative error, whose leading order term is $A\,\rho_\alpha*\theta(x)\,y =: D(x)y$; secondly, the absolute error, whose leading order term is given by $\overline{D}(x,y) := D(x)yn^0(y)$.

We first present some general bounds on the two errors. Notice that $||D||_{L^\infty(\mathbb{R})} \le A||\theta||_{L^\infty(\mathbb{R})}$, meaning $D(x)$ remains limited as $A\to 0$, and so does the relative error for bounded y. In other words, the shape of populations "far from extinction" (A small) when $\varepsilon=0$ is very robust: such species can dampen the perturbation when $|\varepsilon|\neq 0$. As for $\overline{D}$, thanks to (8), we can compute

$$\|\overline{D}\|_{L^\infty(\mathbb{R}^2)} = C(1-A)\|D\|_{L^\infty(\mathbb{R})} \le CA(1-A)||\theta||_{L^\infty(\mathbb{R})}, \tag{148}$$

for some universal constant $C > 0$. Note that, for any x, the maximum of $|\overline{D}(x,\cdot)|$ is attained at $y = \pm A^{-1/2}$, independently of θ. From (148), the absolute error vanishes both far from extinction ($A \to 0$), and close to extinction ($A \to 1$). In the latter case, this is because $||n^0||_{L^\infty(\mathbb{R})}$ itself goes to zero, see (8).

In the sequel, we shall mostly discuss on $D(x)$, which is tied to the relative error and which we call the *deformation*. On the other hand, expansion (147) has the advantage to be uniform in y thanks to (9) and (29), and our numerical explorations will therefore mainly focus on the absolute error $\overline{D}(x,y)$.

Example 25 (Test case). *If $\theta \equiv 1$, then (147) yields $n^\varepsilon(x,y) \approx n^0(y)\,(1 + \varepsilon A y + \cdots)$. On the other hand, in view of equation (7), the solution is explicitly computed as (recall (8) and Proposition 8)*

$$n^\varepsilon(x,y) = n^\varepsilon(y) = n_0(y-\varepsilon) = \eta C_0 e^{-\frac{1}{2}A(y-\varepsilon)^2} = n_0(y) e^{\varepsilon A y - \varepsilon^2 \frac{A}{2}} \approx n_0(y)\,(1 + \varepsilon A y + \cdots)\,.$$

We thus recover that $D(x) \equiv A$.

6.1 Deformation of the steady state under localized perturbation

Example 26 (Localized prototype case). *Consider $\theta(x) := \mathbf{1}_{(-\ell,\ell)}(x)$, with $\ell > 0$. This θ is not continuous but we may consider a smooth compactly supported approximation so it does not matter much for our discussion. From (14), the perturbation is localized so that we only consider $|x| \leq \frac{\ell}{2}$, for which we compute*

$$\rho_\alpha * \theta(x) = \frac{1}{2}\left(\int_{-\ell}^{x} \alpha e^{-\alpha(x-z)} dz + \int_{x}^{\ell} \alpha e^{-\alpha(z-x)} dz\right) = 1 - e^{-\alpha \ell}\cosh(\alpha x).$$

In this case $D(x) = A\left(1 - e^{-\sqrt{2A}\,\ell}\cosh(\sqrt{2A}\,x)\right)$ for $|x| \leq \frac{\ell}{2}$, and

$$C_{A,\ell} := \|D\|_{L^\infty(-\frac{\ell}{2},\frac{\ell}{2})} = A\left(1 - e^{-\sqrt{2A}\,\ell}\right).$$

For a given $\ell > 0$, $A \mapsto C_{A,\ell}$ is increasing on $(0,1)$, $C_{A,\ell} \to 0$ as $A \to 0$, whereas $C_{A,\ell} \to c_\ell := 1 - e^{-\sqrt{2}\,\ell}$ as $A \to 1$. We thus recover the fact that the population can dampen the perturbation "far from extinction" (A small). On the other hand, populations "hardly surviving" (A close to 1) when $\varepsilon = 0$ are more sensitive to the perturbation which they suffer with the coefficient c_ℓ. Notice that letting $\ell \to +\infty$ yields $D(x) \to A$ and we naturally recover the above test case of Example 25.

Example 27 ("Dirac" case). *Consider $\theta(x) = \theta_h(x) := \frac{1}{2h}\mathbf{1}_{(-h,h)}(x)$, with $h > 0$. Again, this θ_h is not continuous, and since $\|\theta_h\|_{L^\infty(\mathbb{R})} \to +\infty$ as $h \to 0$, we expect that $\varepsilon_0 = \varepsilon_0(h)$ provided by Theorem 3 satisfies $\varepsilon_0(h) \to 0$ as $h \to 0$. Nevertheless, we formally obtain*

$$D(x) = D_h(x) \to A\rho_\alpha(x), \text{ as } h \to 0.$$

Therefore, a large variation of the optimal trait on a very small spatial range ($h \to 0$) induces a deformation which is maximal at the singularity (here $x = 0$), and varies like $A^{\frac{3}{2}}$.

6.2 Deformation of the steady state under periodic perturbation

Example 28 (Periodic prototype case). *Consider $\theta(x) := \sin(\frac{x}{\ell})$, with $\ell > 0$, which is $L = 2\pi\ell$-periodic. Then*

$$\rho_\alpha * \theta(x) = \operatorname{Im} \int_{\mathbb{R}} \rho_\alpha(x-z) e^{i\frac{z}{\ell}} dz = \operatorname{Im} e^{i\frac{x}{\ell}} \widehat{\rho_\alpha}\left(\frac{1}{\ell}\right) = \frac{\ell^2\alpha^2}{\ell^2\alpha^2+1}\sin\left(\frac{x}{\ell}\right). \tag{149}$$

In this case

$$D(x) = C_{A,\ell}\,\theta(x), \quad C_{A,\ell} := \frac{2\ell^2 A^2}{2\ell^2 A + 1}. \tag{150}$$

Hence the deformation is proportional to the perturbation $\theta(x)$ itself. Also, for a given $\ell > 0$, $A \mapsto C_{A,\ell}$ is increasing on $(0,1)$, $C_{A,\ell} \to 0$ as $A \to 0$, whereas $C_{A,\ell} \to c_\ell := \frac{2\ell^2}{2\ell^2+1}$ as $A \to 1$. We thus recover *the fact that the population can dampen the perturbation "far from extinction" (A small). On the other hand, populations "hardly surviving" (A close to 1) when $\varepsilon = 0$ are more sensitive to the perturbation which they suffer with the coefficient c_ℓ. Notice also that $c_\ell \to 0$ as $\ell \to 0$ so that rapidly changing environments are rather harmless (in the sense that the deformation is small). On the other hand, $c_\ell \to 1$ as $\ell \to +\infty$ meaning that, in slowly changing environments, populations hardly surviving when $\varepsilon = 0$ fully suffer the perturbation.*

Remark 29 (Influence of L)**.** *Since the deformation $D(x)$ vanishes as $A \to 0$, let us assume here that $A \in (0,1)$ is fixed. We also fix a 1-periodic profile $\tilde{\theta}(x)$ and set $\theta_L(x) := \tilde{\theta}\left(\frac{x}{L}\right)$. We shall highlight how $D_L(x)$, the deformation corresponding to the perturbation $\theta_L(x)$, is affected by L. Firstly, D_L is obviously L-periodic. Then, we have*

$$\tilde{D}_L(x) := A^{-1}D_L(Lx) = (\rho_\alpha * \theta_L)(Lx) = (\rho_{\tilde{\alpha}} * \tilde{\theta})(x), \qquad \tilde{\alpha} := L\alpha = L\sqrt{2A}.$$

When $L \to 0$, one can check that $\tilde{D}_L$ converges uniformly to $\Theta := \int_0^1 \tilde{\theta}(x)dx = \frac{1}{L}\int_0^L \theta_L(x)dx$, so that

$$D_L(x) \to A\Theta, \quad \textit{uniformly as } L \to 0.$$

Note that a deformation $A\Theta$ also corresponds to the deformation assuming $\theta_L(x) \equiv \Theta$, see Example 25. In other words, in a rapidly changing environment, the population is deformed as if the optimal trait was uniformly equal to its average. In particular, if the average is zero, the steady state is not distorted at first order.

On the other hand, as $L \to +\infty$, $\rho_{\tilde{\alpha}}$ serves as an approximation of identity and $||\tilde{D}_L - \tilde{\theta}||_{L^\infty(\mathbb{R})} \to 0$, so that

$$\|D_L - A\theta_L\|_{L^\infty(\mathbb{R})} \to 0, \quad \textit{as } L \to +\infty.$$

Consequently, the deformation is directly proportional to the optimal trait, meaning the population fully suffers from the perturbation. Note that, since $\tilde{\theta}$ is continuous, the profile $\theta_L(x) = \tilde{\theta}\left(\frac{x}{L}\right)$ flattens as $L \to +\infty$. In particular, in the above limit, we could have replaced θ_L by $x \mapsto \frac{1}{2p}\int_{x-p}^{x+p} \theta_L(z)dz$ for any fixed $p > 0$.

We now present some numerics for the periodic prototype case of Example 28. As mentioned above, we are mainly concerned with the *absolute* error

$$E^\varepsilon(x,y) := n^\varepsilon(x,y) - n^0(y) = \varepsilon\overline{D}(x,y) + o(\varepsilon). \tag{151}$$

To compute $n^\varepsilon(x,y)$ numerically, we consider the Cauchy problem with initial data $n^0(y)$, and retain the asymptotics $t \to +\infty$. The steady state $n^\varepsilon(x,y)$ being unique in a neighborhood of $n^0(y)$, one can reasonably assume such an asymptotic state to be $n^\varepsilon(x,y)$. This is confirmed by comparing with the expected theoretical result from Theorem 3, see Figures 1 and 2.

Remark 30 (Absolute error vs. population distribution)**.** *In Figure 1 (and the ones that follow), we represent $\theta(x)$ with a solid, black line. Notice however that this does not correspond to the optimal trait at position x, given by $\varepsilon\theta(x)$ and represented with a dotted line in Figure 1.*

The maximum of the absolute error $|\overline{D}|$ occurs in positions x such that $|D(x)|$ is maximal and with trait $y = \pm y_A := \pm A^{-1/2}$, as mentioned above. As a consequence, at first order, the maximum of $\overline{D}$ occurs at traits $y = \pm y_A$ that do not depend on θ, thus independently of the optimal traits. On the other hand, the positions x where that maximum is attained directly depends on θ through $D(x)$.

Let us underline that this observation concerns the absolute error $E^\varepsilon(x,y)$, but not the population distribution $n^\varepsilon(x,y)$ itself. For the latter, we observe numerically that its maximum remains close to $y = 0$, for $|\varepsilon|$ small enough. Moreover, thanks to (9) and (29), we have

$$\|n^\varepsilon - n^0 - \varepsilon D(x)yn^0(y)\|_Y = o(\varepsilon), \quad \textit{as } \varepsilon \to 0,$$

so that, keeping only the term corresponding to the index $D^\alpha = D_y$ in (31), and looking at $y = 0$, we obtain

$$|n^\varepsilon_y(x,0) - \varepsilon D(x)n^0(0)| = o(\varepsilon), \quad \textit{as } \varepsilon \to 0.$$

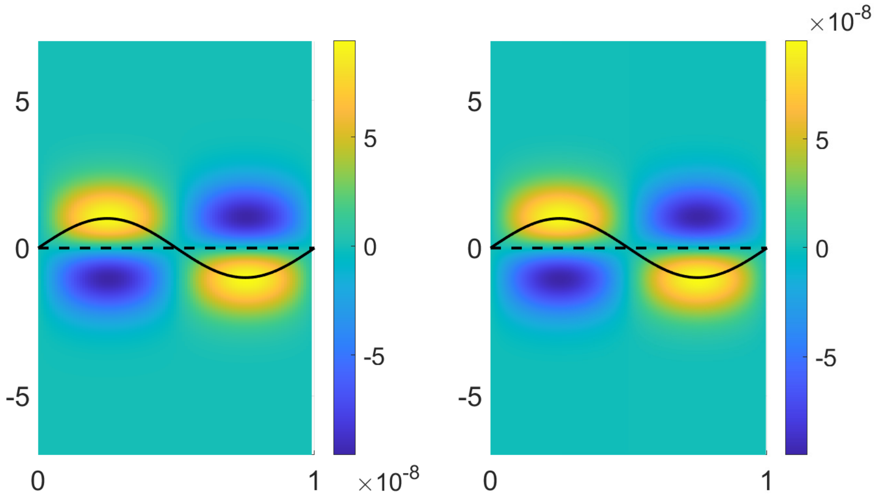

Figure 1: Left: absolute error $E^\varepsilon(x,y) = n^\varepsilon(x,y) - n^0(y)$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical absolute error $\varepsilon\overline{D}(x,y)$. In black, the function $\theta(x) = \sin(2\pi x)$, i.e. $L = 1$. In dotted line, the optimal trait $\varepsilon\theta(x)$. Here $A = 0.9$ and $\varepsilon = 10^{-4}$.

Consequently, for positions x such that $D(x) \neq 0$, we see that, for $|\varepsilon|$ small enough, $n_y^\varepsilon(x,0)$ is non-zero and has same (opposite) sign as $D(x)$ when $\varepsilon > 0$ ($\varepsilon < 0$ respectively). In particular, the maximum of $n^\varepsilon(x,y)$ is not attained for traits $y = 0$. For those x, the maximum of the population size is typically shifted towards the optimal trait. Note that this also applies for non-periodic profile θ.

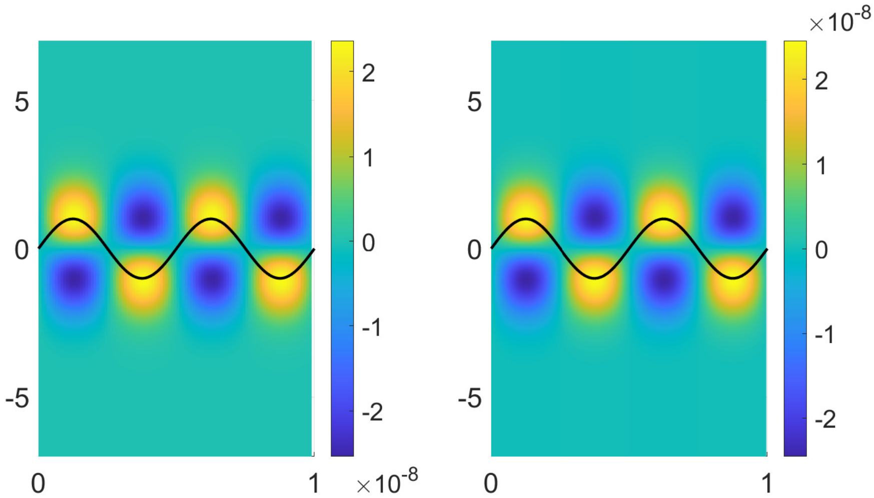

Figure 2: Left: absolute error $E^\varepsilon(x,y) = n^\varepsilon(x,y) - n^0(y)$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical absolute error $\varepsilon\overline{D}(x,y)$. In black, the function $\theta(x) = \sin(4\pi x)$, i.e. $L = 0.5$. Here $A = 0.9$ and $\varepsilon = 10^{-4}$.

Let us pursue with a few comments. Firstly, the error is small near $y = 0$ since $\overline{D}(x,0) = 0$. Also, we see that $E^\varepsilon(x,y)$ has same sign as $y\theta(x)$, since here $D(x,y) = D(x)yn^0(y) = C_{A,\ell}\theta(x)yn^0(y)$. It can be checked that $\|E^\varepsilon - \varepsilon\overline{D}\|_{L^\infty(\mathbb{R}^2)}$ decays numerically like $O(\varepsilon^2)$ as $\varepsilon \to 0$. Let us recall that, from (148),

$$\|\overline{D}\|_{L^\infty(\mathbb{R}^2)} = C(1-A)C_{A,\ell} = C(1-A)\frac{2\ell^2A^2}{2\ell^2A+1},$$

for some universal $C > 0$. Therefore at first order, we expect $E^\varepsilon_{max} := \|E^\varepsilon\|_{L^\infty(\mathbb{R}^2)}$ to be increasing with ℓ, which is highlighted by a comparison of Figures 1 and 2 (notice the different scales). More generally, $\|\overline{D}\|_{L^\infty(\mathbb{R}^2)}$ is “maximal” for $\ell \to +\infty$, $A = \frac{1}{2}$.

Last, we also inquire about the numerical relative error. We refer to Figure 3. We have also computed the relative errors for $A \in \{0.8, 0.9\}$ and $L \in \{0.5, 1\}$. We observed that the numerical outcomes are in agreement with the results discussed in Example 28.

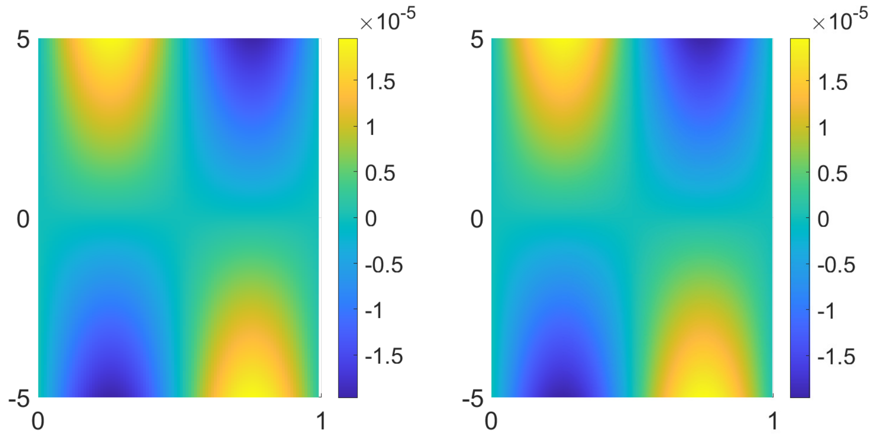

Figure 3: Left: relative error $\frac{n^\varepsilon(x,y)-n^0(y)}{n^0(y)}$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical relative error $\frac{\varepsilon\overline{D}(x,y)}{n^0(y)}$. Here $A = 0.9$, $\varepsilon = 10^{-4}$ and $\theta(x) = \sin(2\pi x)$, i.e. $L = 1$.

Example 31 (Influence of skewness)**.** *We here perform numerical simulations in the 1-periodic step function case*

$$\theta(x) = \begin{cases} +1 & \text{if } x \in \left(0, \frac{a}{2}\right) \cup \left(1 - \frac{a}{2}, 1\right), \\ -1 & \text{if } x \in \left(\frac{a}{2}, 1 - \frac{a}{2}\right), \end{cases} \tag{152}$$

where $0 < a < 1$ serves as a parameter which measures the asymmetry, or skewness, of the perturbation. Indeed, the optimal trait takes the values $y = +\varepsilon$ and $y = -\varepsilon$ with proportions (over a period) a and $1 - a$ respectively.

In the balanced case $a = \frac{1}{2}$, the steady state is symmetrically distorted and, therefore, the location of the maximal absolute error switches between $y = \frac{1}{\sqrt{A}}$ and $y = -\frac{1}{\sqrt{A}}$, see Figure 4.

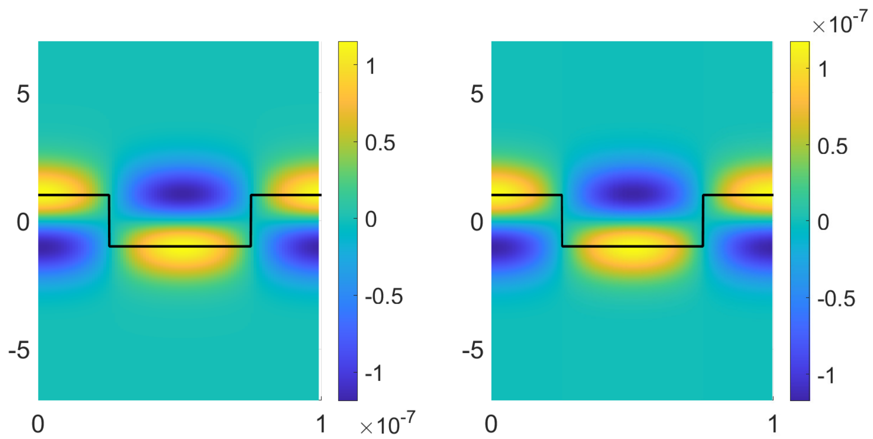

Figure 4: Left: $E^\varepsilon(x,y) = n^\varepsilon(x,y) - n^0(y)$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical $\varepsilon\overline{D}(x,y)$. In black, the 1-periodic function θ given by (152) with $a = 0.5$. Here $A = 0.9$ and $\varepsilon = 10^{-4}$.

On the other hand, when $a \to 1$ (the $a \to 0$ case being similar), the $+\varepsilon$ optimum is much more prevalent and, therefore, there is no switch of the maximal absolute error, and the population leans to the upper side, see Figure 5 for $a = 0.8$. In other words, there is little advantage for the population to invest on displacements to visit the lower side.

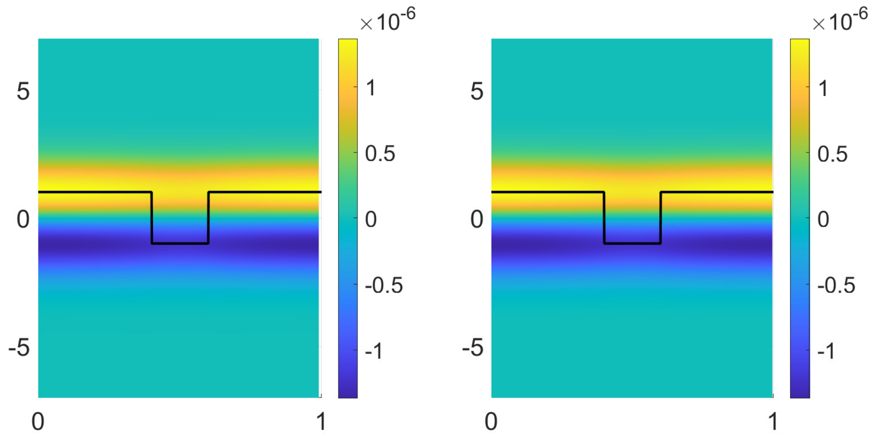

Figure 5: Left: $E^\varepsilon(x,y) = n^\varepsilon(x,y) - n^0(y)$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical $\varepsilon\overline{D}(x,y)$. In black, the 1-periodic function θ given by (152) with $a = 0.8$. Here $A = 0.9$ and $\varepsilon = 10^{-4}$.

Last, we consider an intermediate case: Figure 6, for $a = 0.52$, reveals that the population suffers less from the perturbation at positions x where the optimal trait is $y = -\varepsilon$ than at other positions.

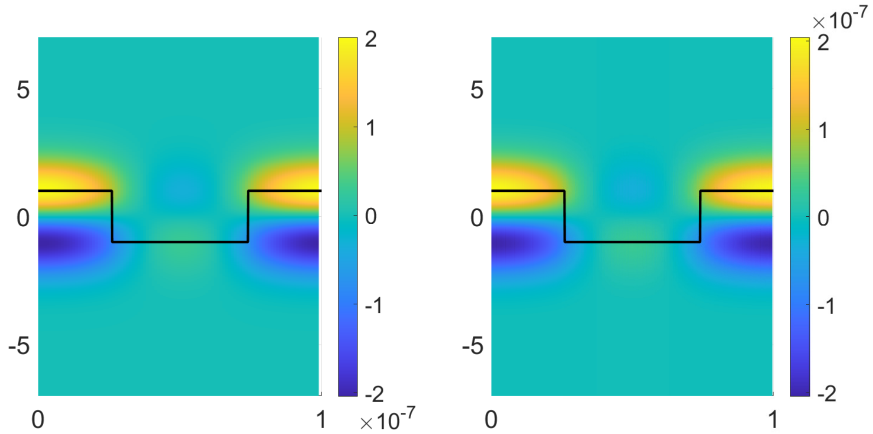

Figure 6: Left: $E^\varepsilon(x,y) = n^\varepsilon(x,y) - n^0(y)$, where $n^\varepsilon(x,y)$ is determined numerically. Right: theoretical $\varepsilon\overline{D}(x,y)$. In black, the 1-periodic function θ given by (152) with $a = 0.52$. Here $A = 0.9$ and $\varepsilon = 10^{-4}$.

It is worth mentioning that Figures 4 to 6 highlight that the maximum absolute error increases (notice the different scales) with $\left|a - \frac{1}{2}\right|$ (i.e. the aforementioned skewness).

These remarks are consistent with the fact that the absolute error is $D(x)yn^0(y) = (\rho_\alpha * \theta)(x)yn^0(y)$. Indeed, in order to have a positive absolute error at the lower side of position x, one must have $D(x) < 0$. In the balanced case, one obviously has $D(x) < 0$ for all $x \in (0.25, 0.75)$, and $|D(x)|$ is maximal at $x = 0.5$. When $a = 0.8$, we have $D(x) \geq D(0.5) > 0$, so that the population always leans towards the upper side, albeit slightly less in $x = 0.5$.

In fact, for any fixed $\alpha = \sqrt{2A}$, one can explicitly compute the value $a = a_\alpha$ such that $D(0.5) = 0$. We omit the details (tedious but straightforward cutting of the integral accordingly to the step function, computation of an infinite series and solving of a quadratic equation) and find

$$a_\alpha = \frac{2}{\alpha} \ln\left(\frac{4}{e^{-\alpha} - 1 + \sqrt{(e^{-\alpha} - 1)^2 + 16e^{-\alpha}}}\right) - 1.$$

Then for any $a > a_\alpha$, we have $D(x) \geq D(0.5) > 0$, so that the population leans to the upper side everywhere. For $A = 0.9$, we have $a_\alpha \approx 0.5273$, hence our choice of $a = 0.52$ for the intermediate case.

6.3 Deformation of the speed and profile of the front under periodic perturbation

Here, we formally reproduce the arguments of subsection 4.3 (performed to analyse the perturbation of the steady state) to analyse the perturbation of the pulsating front constructed through Section 5, to which we refer for notations and definitions. We differentiate $\mathcal{F}^\mu(\varepsilon, s_\varepsilon, v_\varepsilon) = 0$ with respect to ε thanks to the chain rule and then evaluate at $\varepsilon = 0$ to get

$$D_\varepsilon \mathcal{F}^\mu(0,0,0) + \mathcal{L}^\mu \left(\frac{\partial s_{\varepsilon,\mu}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial v_{\varepsilon,\mu}}{\partial \varepsilon}\Big|_{\varepsilon=0} \right) = 0.$$

From the expression of $D_\varepsilon \mathcal{F}^\mu = D_\varepsilon \mathcal{F}^\mu(\varepsilon, s, v)$ and

$$n^\varepsilon(x,y) = n^0(y) + \varepsilon n^1(x,y) + o(\varepsilon) \qquad \text{in } Y, \text{ as } \varepsilon \to 0,$$

we compute

$$f(z,x,y) := D_\varepsilon \mathcal{F}^\mu(0,0,0) = 2U'(z) n^1_x(x,y) + U(z)(1-U(z)) n^0(y) \int_{\mathbb{R}} n^1(x,y')dy' = 2U'(z) n^1_x(x,y),$$

since we know from Theorem 3 that $n^1(x,y)$ is odd with respect to y. From the above and (11), we reach

$$\left(\frac{\partial s_{\varepsilon,\mu}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial v_{\varepsilon,\mu}}{\partial \varepsilon}\Big|_{\varepsilon=0} \right) = (\mathcal{L}^\mu)^{-1}(f) = (\mathcal{L}^\mu)^{-1} \left(\sqrt{2A}\, \eta U'(z) (\rho_A * \theta)'(x) \Gamma_1(y) \right).$$

Projecting on (Γ_j) we thus have $f_j(z,x) = \sqrt{2A}\, \eta U'(z)(\rho_A * \theta)'(x)\, \delta_{j,1}$, where we use the Kronecker symbol. Now, the key point is that $\frac{1}{L}\int_0^L (\rho_A * \theta)'(x)\, dx = 0$ so that the Fourier coefficient $f^0_1(z) \equiv 0$. As a result, recalling (133), $\Phi_f(z) \equiv 0$ so that $s = 0$, where s is given by (135). In our setting, the latter is recast $\frac{\partial s_{\varepsilon,\mu}}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$. Formally letting $\mu \to 0$, this provides $s_\varepsilon = o(\varepsilon)$ and, thus, (19).

As explained above, (19) means that the perturbation of the speed of the front by the nonlinearity $\theta = \theta(x)$ is of the second order with respect to ε. As far as the distortion of the profile of the front itself is involved, we focus on the following example which sheds light on the amplitude of the deformation.

Example 32 (Amplitude of the deformation of the profile)**.** *Here, following Example 28, we consider* $\theta(x) := \sin\left(\frac{x}{\ell}\right)$, *with* $\ell > 0$, *which is* $L = 2\pi\ell$*-periodic. As a result, recalling (11), (12) and (149), we reach*

$$f(z,x,y) = U'(z) \frac{\ell \alpha^4}{\ell^2\alpha^2 + 1} \cos\left(\frac{x}{\ell}\right) y n_0(y) = U'(z) \frac{\ell\alpha^3\eta}{\ell^2\alpha^2+1} \cos\left(\frac{x}{\ell}\right) \Gamma_1(y),$$

where, as above, we use the shortcut $\alpha = \sqrt{2A}$. *Projecting on* (Γ_j), *we get*

$$f_j(z,x) = U'(z) \frac{\ell\alpha^3\eta}{\ell^2\alpha^2+1} \cos\left(\frac{x}{\ell}\right) \delta_{j,1},$$

whose Fourier coefficients are

$$f^n_j(z) = U'(z) \frac{\ell\alpha^3\eta}{2(\ell^2\alpha^2+1)} \delta_{|n|,1}\, \delta_{j,1} =: \mathcal{C}\eta U'(z)\, \delta_{|n|,1}\, \delta_{j,1},$$

and where

$$\mathcal{C} = \mathcal{C}_{A,\ell} := \frac{\ell A\sqrt{2A}}{2\ell^2 A + 1}. \tag{153}$$

In other words $f^1_1(z) = f^{-1}_1(z) = \mathcal{C}\eta U'(z)$ *and all other coefficients vanish. As a result, the profile of the pulsating front is described by*

$$u^\varepsilon(z,x,y) \approx U(z) n^\varepsilon(x,y) + \varepsilon \left(v^{-1}_1(z) e_{-1}(x) + v^1_1(z) e_1(x) \right) \Gamma_1(y) + \cdots,$$

where $\mathcal{E}_{\pm 1,1,\mu}[v^{\pm 1}_1] = f^{\pm 1}_1(z) = \mathcal{C}\eta U'(z)$, *see (100). Clearly, we have* $\overline{v^1_1} = v^{-1}_1$ *so that*

$$\begin{aligned} u^\varepsilon(z,x,y) &\approx U(z) n^\varepsilon(x,y) + \varepsilon\, 2\,\mathrm{Re}\left(v^1_1(z) e_1(x) \right) \Gamma_1(y) + \cdots \\ &\approx U(z) n^0(y) \Big(1 + \varepsilon\, C_{A,\ell} \theta(x) y + \varepsilon\, 2 \frac{\mathrm{Re}\left(v^1_1(z) e_1(x) \right)}{\eta U(z)} \sqrt{2A}\, y + \cdots \Big). \end{aligned}$$

Here we have used Example 28, in particular $C_{A,\ell}$ is given by (150). Next, since $\ell\sqrt{2A}\,\mathcal{C}_{A,\ell} = C_{A,\ell}$, we end up with

$$u^\varepsilon(z,x,y) \approx U(z)n^0(y)\left(1+\varepsilon\, C_{A,\ell}\left(\theta(x)+2\frac{\mathrm{Re}\left(\mathcal{L}^{-1}_{1,1,\mu}[U'](z)e_1(x)\right)}{\ell U(z)}\right)y+\cdots\right). \quad (154)$$

At this stage, since the term $w(z) := \mathcal{L}^{-1}_{1,1,\mu}[U'](z)$ also depends on A and ℓ, the amplitude of (the leading order term of) the deformation of the profile of the front is not transparent. Nevertheless we can formally obtain some clues in some asymptotic regimes. Recall that, up to letting $\mu \to 0$, w solves

$$w'' + \left(\frac{2i}{\ell}+c_0\right)w' - \left(\lambda_1 - \lambda_0 U(z) + \frac{1}{\ell^2}\right)w = U'(z). \quad (155)$$

Letting $\ell \to 0$, (155) formally provides $w(z) \sim -\ell^2 U'(z)$ so that $\frac{\mathrm{Re}\,(w(z)e_1(x))}{\ell U(z)}$ is of "magnitude ℓ", and thus $u^\varepsilon(z,x,y) \approx U(z)n^0(y)\,(1+\varepsilon C_{A,\ell}\theta(x)y+\cdots)$. On the other hand, letting $\ell \to +\infty$, (155) formally shows that $w(z)$ is independent on ℓ so that $\frac{\mathrm{Re}\,(w(z)e_1(x))}{\ell U(z)}$ is of "magnitude $1/\ell$" and thus, again, $u^\varepsilon(z,x,y) \approx U(z)n^0(y)\,(1+\varepsilon C_{A,\ell}\theta(x)y+\cdots)$. As a result, at least in any of the asymptotic regimes $\ell \to 0$, $\ell \to +\infty$, the amplitude of (the leading order term of) the deformation of the profile of the front is again measured by $C_{A,\ell}$, so that the biological insights are similar to those of Example 28.

On the other hand, letting $A \to 0$ or $A \to 1$, $w(z)$ formally becomes independent on A and thus

$$u^\varepsilon(z,x,y) \approx U(z)n^0(y)\Big(1+\varepsilon C_{A,\ell}(\theta(x)+\Psi_\ell(z,x))y+\cdots\Big),$$

so that an additional deformation term, denoted $\Psi_\ell(z,x)$, is involved.

6.4 Numerical support for some conjectures on the pulsating fronts

As mentioned after Theorem 6, the positivity of the pulsating front $u^\varepsilon(z,x,y)$ is not provided by our proof. We nonetheless provide some numerical explorations for the front that, in particular, support its positivity. This task is not straightforward since the first-order term of expansion (154) is not explicit, contrary to the steady state case. Thus we cannot directly compare our numerical front with a theoretical one.

In this subsection, to improve clarity, we consider the front u^ε in the (t,x,y) variables instead of the $(z = x - c_\varepsilon t, x, y)$ variables. From (154), and because $c_\varepsilon = c_0 + o(\varepsilon)$, we have

$$u^\varepsilon(t,x,y) \approx U(x-c_0t)n^0(y)\Big(1+\varepsilon\big(C_{A,\ell}\theta(x)y + \Psi_\ell(x-c_0t,x)y\big)+\dots\Big).$$

Our approach consists in choosing the "explicit part" of the above expansion, that is

$$\bar U(t,x,y) := U(x-c_0t)n^0(y)\Big(1+\varepsilon C_{A,\ell}\theta(x)y\Big),$$

as the "theoretical front". Notice also that, since $\Psi_\ell \to 0$ formally as $\ell \to 0$ or $\ell \to +\infty$ (see the end of subsection 6.3), we expect $u^\varepsilon \approx \bar U$ in those regimes.

Let us recall that U is the Fisher–KPP front that solves (16), with $c_0 \geq c_0^* := 2\sqrt{-\lambda_0}$. For the sake of comparison, we shall consider here $c_0 = 5\sqrt{\frac{-\lambda_0}{6}}$, for which the Fisher–KPP front U, normalized by $U(0)=\frac{1}{2}$, is explicitly known:

$$U(z) = \left(1+(\sqrt{2}-1)\exp\left(\sqrt{\frac{-\lambda_0}{6}}\,z\right)\right)^{-2}.$$

Equipped with the above, we now present our numerical computations, see Figure 7. We first observe that we numerically have $u^\varepsilon > 0$. Also, we check that $u^\varepsilon = u^\varepsilon(t,x,y)$ is indeed a perturbation of $u^0(t,x,y) = U(x-c_0t)n^0(y)$, with the absolute error being maximal *at* the front, leading to a maximal relative error of order similar to $\varepsilon = 10^{-4}$.

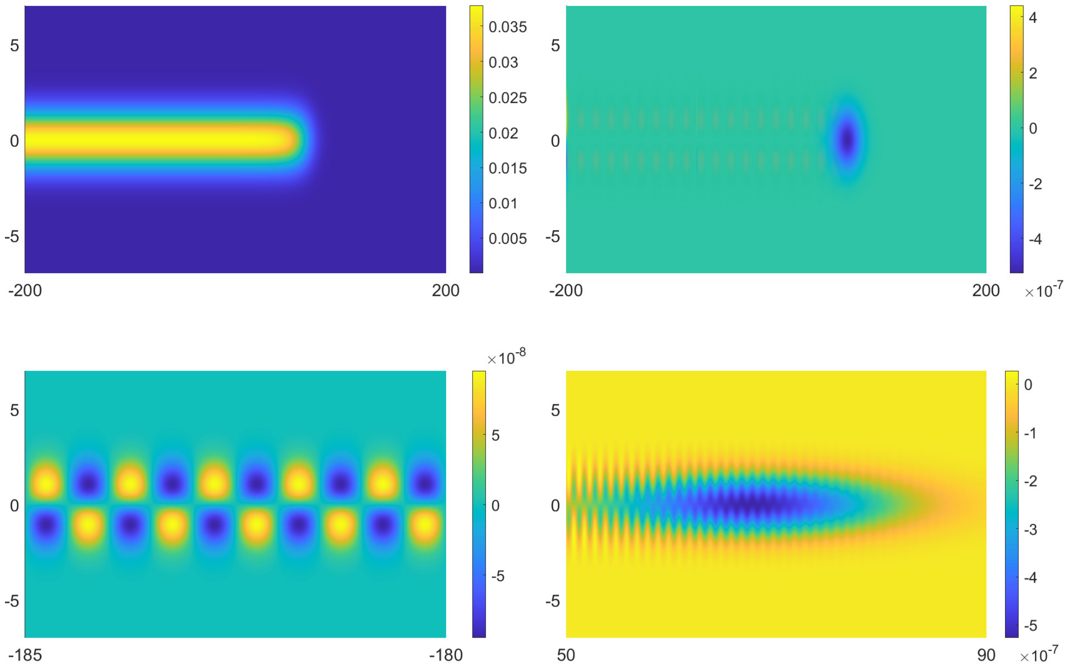

Figure 7: Top left: the front $u^\varepsilon(t = 100, \cdot, \cdot)$ determined numerically. Top right: the error $u^\varepsilon - u^0$ at $t = 100$. Bottom left and right: the same error but with a zoom in on the x-direction. Here $\theta(x) = \sin(2\pi x)$, $A = 0.9$ and $\varepsilon = 10^{-4}$. The initial data is $U(x)n^0(y)$, which, when $\varepsilon = 0$, yields to a "classical" traveling wave solution of speed $c_0 = 5\sqrt{\frac{-\lambda_0}{6}} \approx 0.65$.

However, we notice from Figure 7 that the function $u^\varepsilon - u^0$ presents some small variations behind the front. As long as we look "far" behind the front, the error $u^\varepsilon - u^0$ presents patterns similar, both qualitatively and quantitatively, to Figure 1, which is concerned with the steady state. On the other hand, when looking around the front, we observe that those oscillations disappear. Figure 7 thus suggests that $\bar{U}$ may be a better approximation of u^ε than u^0. The next step is then to compare u^ε and $\bar{U}$, see Figure 8. We see that the small variations behind the front have disappeared, meaning that, at these positions, $\bar{U}$ is a better approximation of u^ε than u^0. Yet, the front location of u^ε, who spreads at speed c_ε, is slightly behind those of u^0 and $\bar{U}$, which spread at speed c_0. This supports the conjecture that $c_\varepsilon < c_0$.

Another natural question is whether or not the numerical solution u^ε is indeed a pulsating front, and if so to determine its propagation speed. The pulsating nature is hard to capture since, on the one hand, there is "a mix of spatial and temporal periodicity" and, on the other hand, those periods depend on c_ε, which is unknown. Nonetheless, from Figure 8, we see that u^ε is very close to the pulsating front $\bar{U}$ apart from positions x around the front. This remains true over a period of time of $\bar{U}$, which provides a partial answer to the pulsating nature. As for the perturbed speed c_ε, it seems impossible to approximate numerically, since we expect $c_\varepsilon - c_0$ to be of order ε^2, and thus the higher-order terms in (154) should be involved in the calculation.

A Appendix

A.1 Proof of Lemma 20

We first need to construct solutions of (104) on $\mathbb{R}_-$ and $\mathbb{R}_+$. The proof mainly consists in rewriting the ordinary differential equation (104) as a fixed point problem, and then to perform careful estimates by considering separately large values of $\min(|n|, j)$ from bounded values of $\min(|n|, j)$.

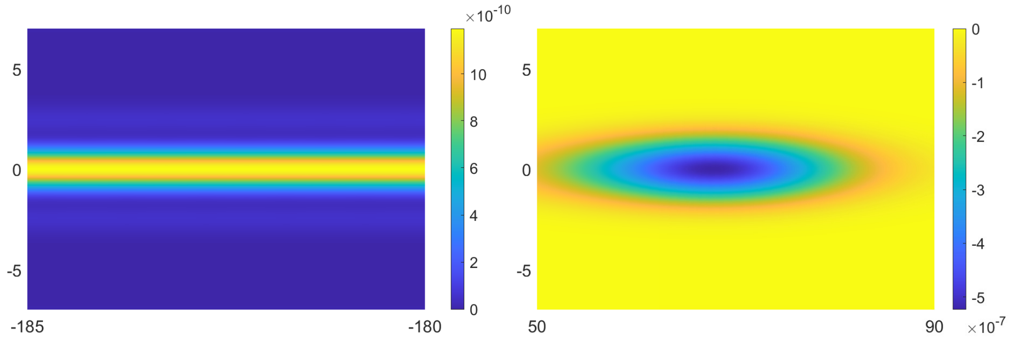

Figure 8: The error $u^\varepsilon - \bar{U}$ at $t = 100$, for different ranges of x. The conditions are the same as in Figure 7.

Lemma 33 (Fundamental system of (104) on $\mathbb{R}_-$ and $\mathbb{R}_+$)**.** *Let $(n, j) \in \mathbb{Z}\times\mathbb{N}$ with $(n, j) \neq (0, 0)$, and $0 \leq \mu < 1$. On $\mathbb{R}_-$, we can construct a system of fundamental solutions $(\tilde{\varphi}_-, \tilde{\varphi}_+)$ of (104), such that*

$$\tilde{\varphi}_\pm(z) = \tilde{P}_\pm(z)e^{a^\pm_{n,j,\mu}z}, \qquad \tilde{P}_\pm \in C^2_b(\mathbb{R}_-, \mathbb{C}), \qquad \liminf_{z\to-\infty}\left|\tilde{P}_-(z)\right| > 0, \tag{156}$$

with $a^\pm_{n,j,\mu}$ given by (107). On $\mathbb{R}_+$, we can construct a system of fundamental solutions $(\tilde{\psi}_-, \tilde{\psi}_+)$ of (104) such that

$$\tilde{\psi}_\pm(z) = \tilde{Q}_\pm(z)e^{b^\pm_{n,j,\mu}z}, \qquad \tilde{Q}_\pm \in C^2_b(\mathbb{R}_+, \mathbb{C}), \qquad \liminf_{z\to+\infty}\left|\tilde{Q}_+(z)\right| > 0, \tag{157}$$

with $b^\pm_{n,j,\mu}$ given by (108). Also, there is $\tilde{R}_{max} > 0$ such that

$$\sup_{(n,j)\neq(0,0)}\ \sup_{0\leq\mu<1}\ \sup_{\tilde{R}\in\{\tilde{P}_\pm,\tilde{Q}_\pm\}}\left(||\tilde{R}||_{L^\infty} + ||\tilde{R}'||_{L^\infty}\right) \leq \tilde{R}_{max}, \tag{158}$$

where by convention the sup norm is taken over the domain of definition of $\tilde{R}$.

Additionally, there exist $n_0, j_0 > 0$ such that if $|n| \geq n_0$ or $j \geq j_0$, there holds for all $0 \leq \mu < 1$,

$$\left|\tilde{P}_+(0) - 1\right|, \left|\tilde{Q}_-(0) - 1\right| \leq \frac{1}{2}, \qquad \tilde{P}_-(0) = \tilde{Q}_+(0) = 1, \tag{159}$$

$$\left|\tilde{P}'_+(0)\right|, \left|\tilde{Q}'_-(0)\right| \leq 1, \qquad \tilde{P}'_-(0) = \tilde{Q}'_+(0) = 0. \tag{160}$$

Besides, denoting $\tilde{P}_\pm = \tilde{P}^{n,j,\mu}_\pm$ and $\tilde{Q}_\pm = \tilde{Q}^{n,j,\mu}_\pm$, we have for any $N_0, J_0 > 0$

$$\sup_{\substack{|n|\leq N_0, j\leq J_0\\(n,j)\neq(0,0)}}\ \sup_{\tilde{R}\in\{\tilde{P}_\pm,\tilde{Q}_\pm\}}\left(\left|\tilde{R}^{n,j,\mu}(0) - \tilde{R}^{n,j,0}(0)\right| + \left|\left(\tilde{R}^{n,j,\mu}\right)'(0) - \left(\tilde{R}^{n,j,0}\right)'(0)\right|\right) \xrightarrow[\mu\to0]{} 0. \tag{161}$$

Next, by taking $\tilde{\mu}_{max} > 0$ small enough, there exists $W_{min} > 0$ such that for all $(n, j) \neq (0, 0)$ and $0 \leq \mu < \tilde{\mu}_{max}$, the Wronskians of $(\tilde{\varphi}_-, \tilde{\varphi}_+)$ and $(\tilde{\psi}_-, \tilde{\psi}_+)$ in zero satisfy:

$$\begin{cases} |W_{\tilde{\varphi}}| := \left|\left[\tilde{\varphi}'_-\tilde{\varphi}_+ - \tilde{\varphi}'_+\tilde{\varphi}_-\right](0)\right| \geq W_{min}, \\ \left|W_{\tilde{\psi}}\right| := \left|\left[\tilde{\psi}'_-\tilde{\psi}_+ - \tilde{\psi}'_+\tilde{\psi}_-\right](0)\right| \geq W_{min}, \end{cases} \tag{162}$$

and if $|n| \geq n_0$ or $j \geq j_0$, we have as well

$$|W_{\tilde{\varphi}}| \geq \frac{1}{4}\left|a^+_{n,j,\mu} - a^-_{n,j,\mu}\right| + 1, \qquad \left|W_{\tilde{\psi}}\right| \geq \frac{1}{4}\left|b^+_{n,j,\mu} - b^-_{n,j,\mu}\right| + 1. \tag{163}$$

Furthermore, there exist $\zeta_1, \zeta_2 > 0$ such that for all $(n, j) \neq (0, 0)$ and $0 \leq \mu < \tilde{\mu}_{max}$

$$\int_{-\infty}^0 |\tilde{\varphi}_+(z)|^2 dz \geq \zeta_1 e^{-\zeta_2 \operatorname{Re} a^+_{n,j,\mu}}. \tag{164}$$

Proof. In the context of this proof, we always assume $(n, j) \neq (0, 0)$. Also, for the sake of readability, we drop the "tilde" notations for $\tilde{\varphi}, \tilde{\psi}, \tilde{P}, \tilde{Q}$ and denote $a^{\pm}_{n,j,\mu} = a^{\pm}$, $b^{\pm}_{n,j,\mu} = b^{\pm}$. We first construct the solutions $\varphi_{\pm}$. Let us fix n, j and $0 \leq \mu < 1$. We only treat the case $j \geq 1$, the proof for $j = 0$ being similar. Set $\varphi_{\pm}(z) = P_{\pm}(z)e^{a^{\pm}z}$ where $P_{\pm} \in C^2_b(\mathbb{R}_-, \mathbb{C})$ is to be determined. Plugging it into (104), we obtain

$$P''_{\pm} \pm rP'_{\pm} - \lambda_0(1 - U(z))P_{\pm} = 0, \tag{165}$$

where $r := 2a^+ + 2in\sigma + c_0 = a^+ - a^-$, so that $\operatorname{Re} r > 0$ from (111).

Let us first construct φ_+. Using a Sturm-Liouville approach, we may recast (165) as

$$(P'_+ e^{rz})' - \lambda_0(1 - U(z))P_+ e^{rz} = 0,$$

so that, assuming $P'_+(-\infty) = 0$, we obtain after integration on $(-\infty, z)$,

$$P'_+(z) = \lambda_0 \int_{-\infty}^{z} e^{r(\omega - z)}(1 - U(\omega))P_+(\omega)d\omega, \tag{166}$$

and thus, assuming $P_+(-\infty) = 1$, after another integration and a ,

$$P_+(z) = 1 - \lambda_0 \int_{-\infty}^{z} \frac{e^{r(\omega - z)} - 1}{r}(1 - U(\omega))P_+(\omega)d\omega. \tag{167}$$

Hence, P_+ is written as the solution of a fixed-point problem. Since $1 - U \in L^1(\mathbb{R}_-)$, for a given $z_0 \leq 0$, the operator in the right-hand side of (167) is globally Lipschitz continuous on $C_b\left((-\infty, z_0], \mathbb{C}\right)$ with Lipschitz constant $2\left|\lambda_0 r^{-1}\right| \int_{-\infty}^{z_0}(1 - U(\omega))d\omega$. Hence, for $|z_0|$ large enough, the fixed-point theorem yields the existence and uniqueness of a solution $P_+ \in C_b\left((-\infty, z_0], \mathbb{C}\right)$ to the problem (167). One can readily check that P_+ indeed solves (165) and belongs to $C^2_b\left((-\infty, z_0], \mathbb{C}\right)$. We extend it to $\mathbb{R}_-$ by solving the Cauchy problem associated to (165). We have therefore constructed a function $\varphi_+(z) = P_+(z)e^{a^+z}$ that solves (104).

We now construct φ_-. We can repeat the same procedure, and obtain that $\varphi_-(z) = P_-(z)e^{a^-z}$ solves (104) if and only if P_- satisfies

$$(P'_- e^{-rz})' - \lambda_0(1 - U(z))P_- e^{-rz} = 0.$$

By integrating on $[z, z_0]$ instead of $(-\infty, z]$, and assuming $P'_-(z_0) = 0$, $P_-(z_0) = 1$, we deduce successively that

$$P'_-(z) = -\lambda_0 \int_{z}^{z_0} e^{-r(\omega - z)}(1 - U(\omega))P_-(\omega)d\omega, \tag{168}$$

$$P_-(z) = 1 + \lambda_0 \int_{z}^{z_0} \frac{1 - e^{-r(\omega - z)}}{r}(1 - U(\omega))P_-(\omega)d\omega, \tag{169}$$

so that P_- solves a fixed-point problem. Assuming $|z_0|$ large enough, there exists a unique solution $P_- \in C_b\left((-\infty, z_0], \mathbb{C}\right)$ by the fixed-point theorem. One can then readily check that $P_- \in C^2_b\left((-\infty, z_0], \mathbb{C}\right)$, and after extending it to $\mathbb{R}_-$ by solving the Cauchy problem associated to (165), we obtain another solution $\varphi_-(z) = P_-(z)e^{a^-z}$ of (104) on $\mathbb{R}_-$. Finally, the solutions (φ_-, φ_+) are linearly independent since $\operatorname{Re} a^+ \neq \operatorname{Re} a^-$.

We shall now prove that there exist $n_0, j_0 > 0$ such that $P_{\pm}$ satisfy (159)—(160) when $|n| \geq n_0$ or $j \geq j_0$. Also, for those indexes n, j, we shall prove that $P_{\pm}$ satisfy (156) and (158). Here we denote $r = r_{n,j,\mu}$. Since (113) and (115) hold, there exist $n_0, j_0 > 0$ such that if $|n| \geq n_0$ or $j \geq j_0$, we have

$$|r_{n,j,\mu}| \geq 8, \quad \frac{|\lambda_0|}{\operatorname{Re} r_{n,j,\mu}} \leq \frac{2}{3}, \quad \left|\frac{\lambda_0}{r_{n,j,\mu}}\right| \int_{-\infty}^{0}(1 - U(\omega))d\omega \leq \frac{1}{6}, \qquad \forall \mu \in [0, 1). \tag{170}$$

Let us assume that $|n| \geq n_0$ or $j \geq j_0$. Then (167) and (169) hold for any $z \leq z_0 = 0$, independently of $0 \leq \mu < 1$. Therefore $P_-(0) = 1$, $P'_-(0) = 0$ and

$$||P_{\pm}||_{C^0_b(\mathbb{R}_-)} \leq 1 + \left(2\left|\frac{\lambda_0}{r_{n,j,\mu}}\right| \int_{-\infty}^{0}(1 - U(\omega))d\omega\right) ||P_{\pm}||_{C^0_b(\mathbb{R}_-)} \leq 1 + \frac{1}{3}||P_{\pm}||_{C^0_b(\mathbb{R}_-)},$$

and thus $||P_\pm||_{C_b^0(\mathbb{R}_-)} \leq \frac{3}{2}$. Combining this bound with (166)— (170), we deduce on the one hand,

$$||P_\pm - 1||_{C_b^0(\mathbb{R}_-)} \leq 2\left|\frac{\lambda_0}{r_{n,j,\mu}}\right| ||P_\pm||_{C_b^0(\mathbb{R}_-)} \int_{-\infty}^0 (1-U(\omega))d\omega \leq \frac{1}{2},$$

and on the other hand,

$$||P'_\pm||_{C_b^0(\mathbb{R}_-)} \leq |\lambda_0| \int_{-\infty}^0 e^{\omega \operatorname{Re} r_{n,j,\mu}} (1-U(\omega)) \, |P_\pm(\omega)| \, d\omega \leq \frac{3\,|\lambda_0|}{2 \operatorname{Re} r_{n,j,\mu}} \leq 1.$$

In conclusion, assuming $|n| \geq n_0$ or $j \geq j_0$, $P_\pm$ satisfy (156) and (158)—(160) with $\tilde{R}_{max} = \frac{5}{2}$.

Let us now fix n, j such that $|n| \leq n_0$ and $j \leq j_0$. We shall prove that, up to taking $\tilde{R}_{max}$ possibly larger, $P_\pm$ satisfy (156) and (158). Note that

$$|r_{n,j,\mu}| \leq |r_{n_0,j_0,1}| =: r_{max} > 0,$$

while we also have, since $(n,j) \neq (0,0)$,

$$|r_{n,j,\mu}| \geq \operatorname{Re} r_{n,j,\mu} \geq \inf_{(n,j)\neq(0,0)} \operatorname{Re} r_{n,j,0} \geq \min(\operatorname{Re} r_{1,0,0}, r_{0,1,0}) =: r_{min} > 0. \tag{171}$$

We now select $z_0 \leq 0$ independent of n, j, μ such that

$$|\lambda_0| \int_{-\infty}^{z_0} (1-U(\omega))d\omega \leq \min\left(\frac{2}{3}, \frac{r_{min}}{6}\right), \tag{172}$$

and thus (167) and (169) hold for any $z \leq z_0$. Similarly as above, we deduce

$$||P_\pm - 1||_{C_b^0((-\infty,z_0))} \leq \frac{1}{2}, \quad ||P'_\pm||_{C_b^0((-\infty,z_0))} \leq 1. \tag{173}$$

From there, we recall that the functions $P_\pm$ are extended to $\mathbb{R}_-$ by solving the Cauchy problem associated to (165), which we recast

$$Y'_\pm = A_\pm(z) Y_\pm, \qquad Y_\pm = \begin{pmatrix} P_\pm \\ P'_\pm \end{pmatrix}, \quad A_\pm(z) = \begin{pmatrix} 0 & 1 \\ \mp r_{n,j,\mu} & \lambda_0(1-U(z)) \end{pmatrix}.$$

If we denote $||\cdot||_\infty$ both the supremum norm on $\mathbb{C}^2$ and its associated subordinate norm on $M_2(\mathbb{C})$, we thus obtain

$$||Y'_\pm(z)||_\infty \leq ||A_\pm(z)||_\infty ||Y_\pm(z)||_\infty \leq M ||Y_\pm(z)||_\infty, \qquad M := \left\|\begin{pmatrix} 0 & 1 \\ r_{max} & |\lambda_0| \end{pmatrix}\right\|_\infty.$$

Using the Gronwall's Lemma, this leads to, for all $z_0 \leq z \leq 0$:

$$||Y_\pm(z)||_\infty \leq ||Y_\pm(z_0)||_\infty e^{M(z-z_0)} \leq \frac{3}{2} e^{M|z_0|}, \qquad \forall |n| \leq n_0, \forall j \leq j_0, \forall \mu \in [0,1).$$

In conclusion, combining this paragraph and the previous one, we deduce that $P_\pm$ satisfy (156) and (158) for $\tilde{R}_{max} = \max\left(\frac{5}{2}, \frac{3}{2}e^{M|z_0|}\right)$. Note that n_0, j_0, z_0 do not depend on μ, so that this is also the case for $\tilde{R}_{max}$.

Let us prove that $W_{\tilde{\varphi}}$ satisfies (163). Given that

$$W_{\tilde{\varphi}} = \left[(a^- - a^+)P_+P_- + P'_-P_+ - P'_+P_-\right](0), \tag{174}$$

and using (159)—(160) with (170), we have, for all $0 \leq \mu < 1$,

$$|W_{\tilde{\varphi}}| \geq \frac{1}{2}\left|a^+ - a^-\right| - 1 \geq \frac{1}{4}|a^+ - a^-| + 1, \qquad \text{if } |n| \geq n_0 \text{ or } j \geq j_0,$$

so that (163) holds.

We now fix $N_0, J_0 > 0$ and show that $P_\pm$ satisfies (161). We first consider fixed indexes $|n| < n_0$ and $j < j_0$. Let us recall that for those n, j we selected $z_0 \le 0$ such that (172) holds for all $0 \le \mu < 1$, which means (167) and (169) hold for any $z \le z_0$. To begin with, we first prove that

$$\sup_{|n|\le n_0, j\le j_0} \left(\left| P_\pm^{n,j,\mu}(z_0) - P_\pm^{n,j,0}(z_0) \right| + \left| \left(P_\pm^{n,j,\mu}\right)'(z_0) - \left(P_\pm^{n,j,0}\right)'(z_0) \right| \right) \xrightarrow[\mu\to 0]{} 0, \tag{175}$$

where we denoted $P_\pm = P_\pm^{n,j,\mu}$. Let us mention that P_- satisfies (175) since by construction $P_-^{n,j,\mu}(z_0) = 1$ and $(P_-^{n,j,\mu})'(z_0) = 0$ for all $0 \le \mu < 1$. It thus suffices to show that $P_+^{n,j,\mu}$ satisfies (175). Fix $\varepsilon > 0$. We set

$$g_{n,j,\mu}(z) := \frac{e^{r_{n,j,\mu} z} - 1}{r_{n,j,\mu}}, \quad 0 \le \mu < 1, \quad z \le 0.$$

Note that, due to (171), we have $||g_{n,j,\mu}||_\infty \le \frac{2}{r_{min}}$. Also, we fix $z_\varepsilon \le z_0$ such that $\int_{-\infty}^{z_\varepsilon}(1-U(\omega))d\omega \le \varepsilon$. Consequently, we have for all $z \le z_0$

$$\begin{aligned}
\left| P_+^{n,j,\mu}(z) - P_+^{n,j,0}(z) \right| \le{}& |\lambda_0| \left| \int_{-\infty}^{z_\varepsilon} \left[(g_{n,j,\mu} - g_{n,j,0})(\omega - z)\right](1 - U(\omega)) P_+^{n,j,\mu}(\omega) d\omega \right| \\
&+ |\lambda_0| \left| \int_{z_\varepsilon}^{z} \left[(g_{n,j,\mu} - g_{n,j,0})(\omega - z)\right](1 - U(\omega)) P_+^{n,j,\mu}(\omega) d\omega \right| \mathbf{1}_{(z_\varepsilon, z_0]}(z) \\
&+ |\lambda_0| \left| \int_{-\infty}^{z} g_{n,j,0}(\omega - z)(1 - U(\omega)) \left[P_+^{n,j,\mu}(\omega) - P_+^{n,j,0}(\omega)\right] d\omega \right| \\
\le{}& 2\varepsilon\, |\lambda_0|\, ||g_{n,j,\mu}||_\infty ||P_+^{n,j,\mu}||_{L^\infty(\mathbb{R}^-)} \\
&+ |\lambda_0|\, ||P_+^{n,j,\mu}||_{L^\infty(\mathbb{R}^-)} \sup_{z_\varepsilon \le \omega \le 0} |(g_{n,j,\mu} - g_{n,j,0})(\omega)| \int_{-\infty}^{0} (1 - U(\omega)) d\omega \\
&+ |\lambda_0|\, ||g_{n,j,0}||_\infty \int_{-\infty}^{z} (1 - U(\omega)) \left| P_+^{n,j,\mu}(\omega) - P_+^{n,j,0}(\omega) \right| d\omega.
\end{aligned}$$

Also, one can readily check that

$$\sup_{|n|<n_0, j<j_0} \sup_{z_\varepsilon \le \omega \le 0} |(g_{n,j,\mu} - g_{n,j,0})(\omega)| \xrightarrow[\mu\to 0]{} 0.$$

Therefore there exists $\mu_\varepsilon > 0$ such that for any $|n| < n_0$, $j < j_0$, $0 \le \mu \le \mu_\varepsilon$ and $z \le z_0$, there holds

$$\left| P_+^{n,j,\mu}(z) - P_+^{n,j,0}(z) \right| \le C\varepsilon + D \int_{-\infty}^{z} (1 - U(\omega)) \left| P_+^{n,j,\mu}(\omega) - P_+^{n,j,0}(\omega) \right| d\omega,$$

$$C := |\lambda_0|\, \tilde{R}_{max} \left(\frac{4}{r_{min}} + \int_{-\infty}^{0} (1 - U(\omega)) d\omega \right) > 0, \qquad D := \frac{2\,|\lambda_0|}{r_{min}} > 0.$$

From the Gronwall's Lemma, we obtain

$$\left| P_+^{n,j,\mu}(z) - P_+^{n,j,0}(z) \right| \le C\varepsilon \exp\left(D \int_{-\infty}^{0} (1 - U(\omega)) d\omega \right).$$

Since ε is arbitrary, we see that $\left| P_+^{n,j,\mu}(z_0) - P_+^{n,j,0}(z_0) \right| \to 0$ as $\mu \to 0$ uniformly in $|n| < n_0$ and $j < j_0$. The proof for $(P_+^{n,j,\mu})'$ is similar and is thus omitted. Therefore (175) holds. We are now ready to prove (161) for indexes $|n| < n_0$ and $j < j_0$. Let us recall that $P_\pm^{n,j,\mu}$ is extended to $\mathbb{R}_-$ by solving the Cauchy problem associated to (165) with initial data taken at $z = z_0$. Because z_0 does not depend on μ, we deduce from classical results of ODEs and continuous dependency of the solutions with respect to the parameter μ, that

$$\sup_{|n|\le n_0, j\le j_0} \left(\left| P_\pm^{n,j,\mu}(0) - P_\pm^{n,j,0}(0) \right| + \left| \left(P_\pm^{n,j,\mu}\right)'(0) - \left(P_\pm^{n,j,0}\right)'(0) \right| \right) \xrightarrow[\mu\to 0]{} 0.$$

We now consider indexes (n,j) such that $n_0 \leq |n| \leq N_0$ or $j_0 \leq j \leq J_0$, assuming $N_0 \geq n_0$ or $J_0 \geq j_0$. Let us recall that for such indexes, (170) holds for all $0 \leq \mu < 1$, which means (167) and (169) hold for any $z \leq 0$. Then using the same arguments, we have that (175) holds where (n_0, j_0, z_0) are replaced by $(N_0, J_0, 0)$, and thus $P_\pm^{n,j,\mu}$ satisfies (161).

We are now ready to prove (162). We first consider fixed indexes $|n| \leq n_0$ and $j \leq j_0$. Then one can readily check that

$$\sup_{|n| \leq n_0, j \leq j_0} |a^\pm_{n,j,\mu} - a^\pm_{n,j,0}| \xrightarrow[\mu \to 0]{} 0.$$

Since (161) holds with $(N_0, J_0) = (n_0, j_0)$, we deduce from (174) that

$$\sup_{|n| \leq n_0, j \leq j_0} \left| W^{n,j,\mu}_{\tilde{\varphi}} - W^{n,j,0}_{\tilde{\varphi}} \right| \xrightarrow[\mu \to 0]{} 0,$$

where we denoted $W_{\tilde{\varphi}} = W^{n,j,\mu}_{\tilde{\varphi}}$. We have $W^{n,j,0}_{\tilde{\varphi}} \neq 0$ for all n, j, since it is the Wronskian of (φ_-, φ_+) when $\mu = 0$. Therefore there exists $m > 0$ such that $\inf_{|n| \leq n_0, j \leq j_0} \left| W^{n,j,0}_{\tilde{\varphi}} \right| \geq m$. Thus taking $\tilde{\mu}_{max} > 0$ small enough, we obtain for any $0 \leq \mu < \tilde{\mu}_{max}$

$$\inf_{|n| \leq n_0, j \leq j_0} \left| W^{n,j,\mu}_{\tilde{\varphi}} \right| \geq \frac{m}{2}.$$

Combining this with (163), we obtain (162) with $W_{min} := \min\left(1, \frac{m}{2}\right)$.

Finally, let us prove (164). Let us consider indexes (n,j) such that $|n| \geq n_0$ or $j \geq j_0$. Then $|P_+(0)| \geq \frac{1}{2}$ from (159). Set $\rho = (4\tilde{R}_{max})^{-1} > 0$. Then by the mean value inequality we have for all $\mu \in [0,1)$

$$\int_{-\infty}^0 |\varphi_+(z)|^2 \, dz \geq \int_{-\rho}^0 |P_+(z)|^2 e^{2 \operatorname{Re} a^+ z} dz \geq \left(\frac{1}{2} - \tilde{R}_{max} \rho \right)^2 \int_{-\rho}^0 e^{2 \operatorname{Re} a^+ z} dz \geq \frac{\rho}{16} e^{-2 \operatorname{Re} a^+ \rho}.$$

Let us now assume that $|n| \leq n_0$ and $j \leq j_0$. Then from (173) we have $|P_+(z_0)| \geq \frac{1}{2}$, with $z_0 \leq 0$ independent of n, j, μ, see (172). Redoing the same calculations with $\int_{-\rho}^0$ being replaced by $\int_{z_0 - \rho}^{z_0}$, we obtain

$$\int_{-\infty}^0 |\varphi_+(z)|^2 \, dz \geq \frac{\rho}{16} e^{2 \operatorname{Re} a^+ (z_0 - \rho)}.$$

Combining those estimates, we obtain (164).

As for the construction of solutions (ψ_-, ψ_+) on $\mathbb{R}_+$, the procedure is similar and leads to the construction of a system of fundamental solutions (ψ_-, ψ_+) of (104) on $\mathbb{R}_+$ such that (157)—(163) hold. Let us however underline a key difference: there may happen that $\operatorname{Re} b^+ \leq 0$ since (110) holds. Despite that, $r := 2b^+ + 2in\sigma + c_0 = b^+ - b^-$ still satisfies $\operatorname{Re} r > 0$ from (111). If however $(n, j, c_0) = (0, 0, c_0^*)$, then $r = 0$ and the above proof does not work, which is why we excluded the case $n = j = 0$. ☐

We are now in the position to prove Lemma 20 concerning a fundamental system of solutions to (104) on $\mathbb{R}$.

Proof of Lemma 20. In the context of this proof, we always assume $(n,j) \neq (0,0)$, and for the sake of readability, we denote $a^\pm_{n,j,\mu} = a^\pm$, $b^\pm_{n,j,\mu} = b^\pm$. From Lemma 33, for all n, j and $0 \leq \mu < 1$, we are equipped with the functions

$$\tilde{\varphi}_\pm(z) = \tilde{P}_\pm(z) e^{a^\pm z}, \qquad \tilde{\psi}_\pm(z) = \tilde{Q}_\pm(z) e^{b^\pm z},$$

that we extend to $\mathbb{R}$ by solving the Cauchy problem associated to (104). Let us fix n, j and $0 \leq \mu < \tilde{\mu}_{max}$, where $\tilde{\mu}_{max}$ is obtained from Lemma 33. Let us first assume that (n, j, μ) are such that $\tilde{\varphi}_+, \tilde{\psi}_-$ are linearly independent. Then we set $(\varphi_-, \varphi_+) := (\tilde{\psi}_-, \tilde{\varphi}_+)$, which is indeed a fundamental system of solutions. In particular, there exist $c_- \in \mathbb{C}$ and $c_+ \in \mathbb{C} \backslash \{0\}$ such that for any $z \geq 0$, there holds

$$\varphi_+(z) = c_- \tilde{\psi}_-(z) + c_+ \tilde{\psi}_+(z) = c_- \tilde{Q}_-(z) e^{b^- z} + c_+ \tilde{Q}_+(z) e^{b^+ z}.$$

Setting

$$P_+(z) = \tilde{P}_+(z), \qquad Q_+(z) = c_-\tilde{Q}_-(z)e^{(b^- - b^+)z} + c_+\tilde{Q}_+(z), \tag{176}$$

yields that φ_+ satisfies (117) with $P_+ \in C_b^2(\mathbb{R}_-)$ and $Q_+ \in C_b^2(\mathbb{R}_+)$, thanks to (111) and (156)—(157). Also, since $c_+ \neq 0$, we have $\liminf_{z\to+\infty} |Q_+(z)| \geq |c_+| \liminf_{z\to+\infty} |\tilde{Q}_+(z)| > 0$ from (157). We can prove in the same way that φ_- satisfies (117) with P_-, Q_- belonging to $C_b^2(\mathbb{R}_-), C_b^2(\mathbb{R}_+)$ respectively, and $\liminf_{z\to-\infty} |P_-(z)| > 0$.

Next, we claim that $\tilde{\varphi}_+, \tilde{\psi}_-$ are never collinear. Let us assume by contradiction that there exist $(n, j) \neq (0, 0)$ and $0 \leq \mu < \tilde{\mu}_{max}$ such that $\tilde{\varphi}_+, \tilde{\psi}_-$ are collinear. Then adapting the above proof yields that $(\psi_\infty, \psi_0) := (\tilde{\varphi}_-, \tilde{\varphi}_+)$ is a fundamental system of solutions of (104) such that

$$\psi_\infty(z) = \begin{cases} P_-(z)e^{a^- z} & z \leq 0, \\ Q_+(z)e^{b^+ z} & z \geq 0, \end{cases} \qquad \psi_0(z) = \begin{cases} P_+(z)e^{a^+ z} & z \leq 0, \\ Q_-(z)e^{b^- z} & z \geq 0, \end{cases} \tag{177}$$

with $P_\pm \in C_b^2(\mathbb{R}_-)$, $Q_\pm \in C_b^2(\mathbb{R}_+)$, $\liminf_{z\to-\infty} |P_-(z)| > 0$ and $\liminf_{z\to+\infty} |Q_+(z)| > 0$. Next, set $0 < \delta < 1$, and let $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ be the operator defined by (189), with $\rho = \frac{c_0}{2}$. We shall prove that $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ is surjective. For any $f \in C_\rho^{0,\delta}(\mathbb{R}, \mathbb{C})$, using the variation of the constant, we set

$$v(z) := \psi_\infty(z) \int_{-\infty}^z \frac{1}{W(\omega)} \psi_0(\omega) f(\omega) d\omega - \psi_0(z) \int_0^z \frac{1}{W(\omega)} \psi_\infty(\omega) f(\omega) d\omega,$$

with the Wronskian $W(\omega) := [\psi_\infty' \psi_0 - \psi_0' \psi_\infty](\omega) \neq 0$. By construction, $v \in C^2(\mathbb{R}, \mathbb{C})$ and satisfies $\mathcal{E}_{n,j,\mu}[v] = f$. Thus to conclude, it suffices to prove that $v \in C_\rho^{2,\delta}(\mathbb{R}, \mathbb{C})$, or equivalently that $\overline{v}(z) := v(z)e^{\rho z} \in C^{2,\delta}(\mathbb{R}, \mathbb{C})$. Firstly, we clearly have $\overline{v} \in C^2(\mathbb{R}, \mathbb{C})$. Also, notice that since (ψ_∞, ψ_0) solve (104), there holds

$$W(\omega) = W(0)e^{-(c_0 + 2in\sigma)\omega} = W(0)e^{(a^+ + a^-)\omega} = W(0)e^{(b^+ + b^-)\omega}.$$

Let us prove that $\overline{v} \in C_b^2(\mathbb{R}, \mathbb{C})$. Setting $\overline{f} := fe^{\rho z} \in C^{0,\delta}(\mathbb{R}, \mathbb{C})$, for any $z \leq 0$, there holds

$$\begin{aligned}
|\overline{v}(z)| &= \left| P_-(z)e^{(a^- + \rho)z} \int_{-\infty}^z \frac{P_+(\omega)}{W(0)} e^{-(a^- + \rho)\omega} \overline{f}(\omega) d\omega + P_+(z)e^{(a^+ + \rho)z} \int_z^0 \frac{P_-(\omega)}{W(0)} e^{-(a^+ + \rho)\omega} \overline{f}(\omega) d\omega \right| \\
&\leq \frac{1}{|W(0)|} ||P_-||_\infty ||P_+||_\infty ||\overline{f}||_\infty \left(e^{(\mathrm{Re}\ a^- + \rho)z} \int_{-\infty}^z e^{-(\mathrm{Re}\ a^- + \rho)\omega} d\omega + e^{(\mathrm{Re}\ a^+ + \rho)z} \int_z^0 e^{-(\mathrm{Re}\ a^+ + \rho)\omega} d\omega \right) \\
&\leq \frac{1}{|W(0)|} ||P_-||_\infty ||P_+||_\infty ||\overline{f}||_\infty \left(-\frac{1}{\mathrm{Re}\ a^- + \rho} + \frac{1 - e^{(\mathrm{Re}\, a^+ + \rho)z}}{\mathrm{Re}\ a^+ + \rho} \right) \\
&\leq \frac{1}{|W(0)|} ||P_-||_\infty ||P_+||_\infty ||\overline{f}||_\infty \left(-\frac{1}{\mathrm{Re}\ a^- + \rho} + \frac{1}{\mathrm{Re}\ a^+ + \rho} \right),
\end{aligned}$$

where the last two lines of calculation are valid since $\mathrm{Re}\, a^\pm + \rho = \pm\frac{1}{2} \mathrm{Re}(a^+ - a^-)$ and (111) holds. Therefore $\overline{v}$ is uniformly bounded on $\mathbb{R}_-$, and similarly on $\mathbb{R}_+$. Next, because

$$\overline{v}'(z) = \psi_\infty'(z) \int_{-\infty}^z \frac{1}{W(\omega)} \psi_0(\omega) f(\omega) d\omega - \psi_0'(z) \int_0^z \frac{1}{W(\omega)} \psi_\infty(\omega) f(\omega) d\omega,$$

and $P_\pm, Q_\pm \in C_b^1(\mathbb{R}, \mathbb{C})$, we prove in the same way that $\overline{v} \in C_b^1(\mathbb{R}, \mathbb{C})$. Finally, plugging $v = \overline{v}e^{-\rho z}$ into $\mathcal{E}_{n,j,\mu}[v] = f$, we deduce that

$$\overline{\mathcal{E}}_{n,j,\mu}[\overline{v}] := \overline{v}'' + 2in\sigma\overline{v}' + \left[-\frac{c_0^2}{4} - \left(\lambda_j - (1 + \delta_{0j})\lambda_0 U(z) + (1 + \mu)n^2\sigma^2 \right) \right] \overline{v} = \overline{f}.$$

Therefore $\overline{v} \in C_b^2(\mathbb{R}, \mathbb{C})$, and since $\overline{f} \in C^{0,\delta}(\mathbb{R}, \mathbb{C})$, we deduce immediately that $\overline{v} \in C^{2,\delta}(\mathbb{R}, \mathbb{C})$, so that $v \in C_\rho^{2,\delta}(\mathbb{R}, \mathbb{C})$. Therefore $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ is surjective. From Lemma 35 in Appendix A.2, this operator is also Fredholm of index zero, and is thus injective. We shall obtain a contradiction by showing that $\psi_0 \in \ker \mathcal{L}_{n,j,\mu}^{\delta,\rho}$. First, setting $\overline{\psi}_0(z) := \psi_0(z)e^{\rho z}$, we have $\overline{\psi}_0 \in C_b^2(\mathbb{R}, \mathbb{C})$ from (177). Then, since

$\overline{\mathcal{E}}_{n,j,\mu}[\overline{\psi}_0] = 0$, we also have $\overline{\psi}_0 \in C^{2,\delta}(\mathbb{R},\mathbb{C})$. Therefore $\psi_0 \in C^{2,\delta}_\rho(\mathbb{R},\mathbb{C})$ satisfies $\mathcal{E}_{n,j,\mu}[\psi_0] = 0$, which means $\psi_0 \in \ker \mathcal{L}^{\delta,\rho}_{n,j,\mu}$. Consequently our assumption that $\tilde{\varphi}_+, \tilde{\psi}_-$ are collinear is absurd. Therefore for all $(n,j) \neq (0,0)$ and $0 \leq \mu < \tilde{\mu}_{max}$, the functions (φ_-, φ_+) defined by (117) form a fundamental system of solutions of (104).

Let us prove that $P_\pm, Q_\pm$ satisfy (118). We shall only prove it for P_+, Q_+, the equivalent for P_-, Q_- being similar. Since $P_+ = \tilde{P}_+$ satisfies (158), P_+ satisfies (118) if $R_{max} \geq \tilde{R}_{max}$. As for Q_+, we first need upper bounds on $c_\pm$. Since φ_+, φ_+' are continuous at $z = 0$, we obtain from (117) and (176), the following linear system in (c_-, c_+)

$$\begin{cases} c_-\tilde{Q}_-(0) + c_+\tilde{Q}_+(0) & = \tilde{P}_+(0), \\ c_-\left(\tilde{Q}_-'(0) + b^-\tilde{Q}_-(0)\right) + c_+\left(\tilde{Q}_+'(0) + b^+\tilde{Q}_+(0)\right) & = \tilde{P}_+'(0) + a^+\tilde{P}_+(0). \end{cases} \tag{178}$$

Let us recall that we denoted $W_{\tilde{\psi}}$ the Wronskian in $z = 0$ of the family $(\tilde{\psi}_-, \tilde{\psi}_+)$, and there holds (162) since $\mu < \tilde{\mu}_{max}$. Therefore the system (178) admits a unique solution for all n, j and $0 \leq \mu < \tilde{\mu}_{max}$, given by

$$c_- = -\frac{\left(\tilde{Q}_+'(0) + b^+\tilde{Q}_+(0)\right)\tilde{P}_+(0) - \left(\tilde{P}_+'(0) + a^+\tilde{P}_+(0)\right)\tilde{Q}_+(0)}{W_{\tilde{\psi}}}, \tag{179}$$

$$c_+ = -\frac{\left(\tilde{P}_+'(0) + a^+\tilde{P}_+(0)\right)\tilde{Q}_-(0) - \left(\tilde{Q}_-'(0) + b^-\tilde{Q}_-(0)\right)\tilde{P}_+(0)}{W_{\tilde{\psi}}}. \tag{180}$$

Let $n_0, j_0 > 0$ being given by Lemma 33 and fix n, j such that $|n| \geq n_0$ or $j \geq j_0$. Then we deduce from (116), (159)—(160) and (163) that

$$|c_-| \leq \frac{\frac{3}{2}\,|b^+ - a^+| + 1}{\frac{1}{4}|b^+ - b^-| + 1} \leq \frac{6\overline{C} + 4}{|b^+ - b^-| + 4},$$

$$|c_+| \leq \frac{\frac{9}{4}\,|a^+ - b^-| + 3}{\frac{1}{4}|b^+ - b^-| + 1} \leq \frac{9\left(|b^+ - b^-| + \overline{C}\right) + 12}{|b^+ - b^-| + 4}.$$

Thus there exists $C > 0$ such that for all n, j and $0 \leq \mu < \tilde{\mu}_{max}$,

$$|c_-| \leq \frac{C}{1 + |b^+ - b^-|}, \qquad |c_+| \leq C, \qquad \text{if } |n| \geq n_0 \text{ or } j \geq j_0. \tag{181}$$

This leads to, thanks to (158) and (176),

$$||Q_+||_{L^\infty(\mathbb{R}_+)} \leq 2C\tilde{R}_{max}, \qquad \text{if } |n| \geq n_0 \text{ or } j \geq j_0, \tag{182}$$

$$||Q_+'||_{L^\infty(\mathbb{R}_+)} \leq \frac{C\left(\tilde{R}_{max} + |b^+ - b^-|\tilde{R}_{max}\right)}{1 + |b^+ - b^-|} + C\tilde{R}_{max} \leq 2C\tilde{R}_{max}, \qquad \text{if } |n| \geq n_0 \text{ or } j \geq j_0, \tag{183}$$

so that Q_+ satisfies (118) for any such n, j. We now consider fixed indexes $|n| \leq n_0$ and $j \leq j_0$. It is clear that there exists $M > 0$ such that

$$\max\left(|b^\pm_{n,j,\mu}|, |a^\pm_{n,j,\mu}|\right) \leq M, \qquad \forall |n| \leq n_0, \quad \forall j \leq j_0, \quad \forall 0 \leq \mu < \tilde{\mu}_{max}.$$

Then combining this with (158) and (162), we have

$$|c_\pm| \leq \frac{2\,(M+1)\,\tilde{R}^2_{max}}{W_{min}} =: K,$$

so that we deduce

$$||Q_+||_{L^\infty(\mathbb{R}_+)} \leq 2K\tilde{R}_{max}, \qquad \forall |n| \leq n_0, \forall j \leq j_0. \tag{184}$$

$$||Q_+'||_{L^\infty(\mathbb{R}_+)} \leq K\left(\tilde{R}_{max} + |b^+ - b^-|\tilde{R}_{max}\right) + K\tilde{R}_{max} \leq 2K\,(1+M)\,\tilde{R}_{max}, \qquad \forall |n| \leq n_0, \forall j \leq j_0. \tag{185}$$

Consequently, from (182)—(185), we obtain that Q_+ satisfies (118) for all n, j and $0 \leq \mu < \tilde{\mu}_{max}$.

Let us now prove that W_φ satisfies (120). Let us consider indexes n, j such that $|n| \geq N_0 \geq n_0$ *or* $j \geq J_0 \geq j_0$, where $N_0, J_0 > 0$ are large enough so that, thanks to (113), we have

$$\left| b^+_{n,j,\mu} - b^-_{n,j,\mu} \right| \geq \max\left(2\overline{C} + 24\,,\, 702\left(2R^2_{max} + 1\right)\right), \quad \forall \mu \in [0, 1), \tag{186}$$

$$\frac{\frac{1}{12}\left| b^+_{n,j,\mu} - b^-_{n,j,\mu} \right| - \frac{3}{2}C}{\left| b^+_{n,j,\mu} - b^-_{n,j,\mu} \right| + 1} \geq \frac{1}{13}, \quad \forall \mu \in [0, 1). \tag{187}$$

Since $N_0 \geq n_0$ and $J_0 \geq j_0$, we also have thanks to (159)—(160)

$$\left| W_{\tilde{\psi}} \right| \leq \frac{3}{2}|b^+ - b^-| + 1 \leq \frac{3}{2}\left(1 + |b^+ - b^-|\right),$$

and thus, using (116), (159)—(160) and (180), we deduce

$$|c_+| \geq \frac{\frac{1}{4}|b^+ - b^-| - \frac{1}{4}\overline{C} - 3}{\frac{3}{2}\left(1 + |b^+ - b^-|\right)} \geq \frac{\frac{1}{12}|b^+ - b^-|}{1 + |b^+ - b^-|},$$

where we used (186) for the last inequality. Combining this lower bound with (159), (176) and (181), we have

$$|Q_+(0)| \geq |c_+| - \frac{3}{2}|c_-| \geq \frac{1}{13},$$

where we used (187) for the last inequality. Let us also recall that, by construction, we have $|Q_-(0)| = |\tilde{Q}_-(0)| \geq \frac{1}{2}$ from (159) and because $|n| \geq N_0 \geq n_0$. Combining that with (118), we obtain

$$\begin{aligned} W_\varphi &:= \left[(b^- - b^+)Q_-Q_+ + Q'_-Q_+ - Q'_+Q_-\right](0) \\ |W_\varphi| &\geq \frac{1}{26}\left|b^+ - b^-\right| - 2R^2_{max} \\ &\geq \frac{1}{27}\left|b^+ - b^-\right| + 1, \end{aligned} \tag{188}$$

since (186) holds. Thanks to (113), one can readily show that W_φ satisfies (120) with $C_W = \min\left(1, \frac{1}{27}\underline{C}\right)^{-1}$.

Let us prove that $W_\varphi = W^{n,j,\mu}_\varphi$ satisfies (119) for indexes $|n| \leq N_0$ and $j \leq J_0$. From (188), since $Q_- = \tilde{Q}_-$ by construction, since Q_+ satisfies (176) with $c_\pm$ given by (179)—(180), and because $W_{\tilde{\psi}}$ is given by (162), there exists a polynomial function $\mathcal{P}$ such that for all n, j, μ, we have

$$W^{n,j,\mu}_\varphi = \frac{1}{W^{n,j,\mu}_{\tilde{\psi}}}\mathcal{P}\left(\tilde{Q}_-(0), \tilde{Q}'_-(0), \tilde{Q}_+(0), \tilde{Q}'_+(0), \tilde{P}_+(0), \tilde{P}'_+(0), a^+, b^+, b^-\right) =: \frac{1}{W^{n,j,\mu}_{\tilde{\psi}}}\mathcal{P}^{n,j,\mu}.$$

One can readily check that $\left| a^\pm_{n,j,\mu} - a^\pm_{n,j,0} \right|, \left| b^\pm_{n,j,\mu} - b^\pm_{n,j,0} \right| \to 0$ as $\mu \to 0$ uniformly in $|n| \leq N_0$ and $j \leq J_0$. Combining this with (161), we obtain

$$\sup_{|n| \leq N_0, j \leq J_0} \left| \mathcal{P}^{n,j,\mu} - \mathcal{P}^{n,j,0} \right| \xrightarrow[\mu \to 0]{} 0.$$

Meanwhile, the same holds for $W^{n,j,\mu}_{\tilde{\psi}}$, as we showed in the proof of (162). Since (162) holds, we have

$$\sup_{|n| \leq N_0, j \leq J_0} \left| W^{n,j,\mu}_\varphi - W^{n,j,0}_\varphi \right| \xrightarrow[\mu \to 0]{} 0.$$

Also, there holds $W^{n,j,0}_\varphi \neq 0$ for all n, j since φ_-, φ_+ are linearly independent. Thus there exists $m > 0$ such that

$$\inf_{|n| \leq N_0, j \leq J_0} \left| W^{n,j,0}_\varphi \right| \geq m.$$

Therefore, and because N_0, J_0 are uniform in $\mu \in [0,1)$ from (186)—(187), we deduce the existence of $0 < \mu_{max} \leq \tilde{\mu}_{max}$ such that for all $0 \leq \mu < \mu_{max}$ we have

$$\inf_{|n| \leq N_0, j \leq J_0} \left|W_{\varphi}^{n,j,\mu}\right| \geq \frac{m}{2}.$$

Combining this with (120), we see that (119) holds with $W_0 = \min(\frac{1}{C_W}, \frac{m}{2})$.

Finally, (121) simply follows from (164) since $\varphi_+ = \tilde{\varphi}_+$ by construction. The proof of Lemma 20 is thus complete. □

A.2 Fredholm analysis

We recall below [45, Theorem 2.4, p. 366].

Theorem 34 (Fredholm property in $C^{2,\delta}(\mathbb{R}, \mathbb{R}^d)$)**.** *Fix $0 < \delta < 1$ and $d \in \mathbb{N}_+$. Consider the operator $L\colon C^{2,\delta}(\mathbb{R}, \mathbb{R}^d) \to C^{0,\delta}(\mathbb{R}, \mathbb{R}^d)$ defined by*

$$Lu := \alpha(x)u'' + \beta(x)u' + \gamma(x)u,$$

where the coefficients $\alpha(x), \beta(x), \gamma(x)$ are smooth $d \times d$ matrices and there is $\alpha_0 > 0$ such that $\langle \alpha(x)\xi, \xi \rangle \geq \alpha_0 |\xi|^2$ for any $\xi \in \mathbb{R}^d$. Assume that $\alpha(x), \beta(x), \gamma(x)$ have finite limits as $x \to \pm\infty$, denoted respectively $\alpha_\pm, \beta_\pm, \gamma_\pm$. Finally, we define the limiting operators

$$L^{\pm}u = \alpha_\pm u'' + \beta_\pm u' + \gamma_\pm u,$$

and assume that

$$\forall \xi \in \mathbb{R}, \quad T^{\pm}(\xi) = -\alpha_\pm \xi^2 + \beta_\pm i\xi + \gamma_\pm \quad \textit{is an invertible matrix.}$$

Then L is a Fredholm operator, and its index is given by $\operatorname{ind} L = k_+ - k_-$*, where*

$$k_\pm = Sp(M^\pm) \cap \{z \in \mathbb{C} : \operatorname{Re} z > 0\}, \quad M^\pm = \begin{pmatrix} 0 & -I_d \\ \alpha_\pm^{-1}\gamma_\pm & \alpha_\pm^{-1}\beta_\pm \end{pmatrix} \in M_{2d \times 2d}(\mathbb{R}).$$

We now apply Theorem 34 to our case.

Lemma 35 (Fredholm property on weighted spaces)**.** *Let $(n,j) \in \mathbb{Z} \times \mathbb{N}$ with $(n,j) \neq (0,0)$, and $0 \leq \mu < 1$. Set $0 < \delta < 1$, $\rho := \frac{c_0}{2} > 0$ and*

$$\begin{aligned} \mathcal{L}_{n,j,\mu}^{\delta,\rho}\colon C_\rho^{2,\delta}(\mathbb{R}, \mathbb{C}) &\to C_\rho^{0,\delta}(\mathbb{R}, \mathbb{C}) \\ u &\mapsto \mathcal{E}_{n,j,\mu}[u], \end{aligned} \tag{189}$$

where $\mathcal{E}_{n,j,\mu}[u]$ is given by (100), and for any $k \in \mathbb{N}$, we set

$$C_\rho^{k,\delta}(\mathbb{R}, \mathbb{C}) := \left\{ f \in C^k(\mathbb{R}, \mathbb{C}) : ||f||_{C_\rho^{k,\delta}(\mathbb{R},\mathbb{C})} < \infty \right\}, \qquad ||f||_{C_\rho^{k,\delta}(\mathbb{R},\mathbb{C})} := \| z \mapsto f(z)e^{\rho z} \|_{C^{k,\delta}(\mathbb{R},\mathbb{C})}.$$

Then $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ is Fredholm with index zero.

Proof. We cannot apply Theorem 34 to $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ because of the weighted space and the functions involved being complex. Thus we define successively the real counterpart of $\mathcal{L}_{n,j,\mu}^{\delta,\rho}$ by

$$\begin{aligned} L_{n,j,\mu}^{\delta,\rho}\colon C_\rho^{2,\delta}(\mathbb{R}, \mathbb{R}^2) &\to C_\rho^{0,\delta}(\mathbb{R}, \mathbb{R}^2) \\ u = (u_1, u_2) &\mapsto \begin{pmatrix} u_1'' + c_0 u_1' - 2n\sigma u_2' - \left(\lambda_j - (1+\delta_{0j})\lambda_0 U(z) + (1+\mu)n^2\sigma^2\right) u_1 \\ u_2'' + c_0 u_2' + 2n\sigma u_1' - \left(\lambda_j - (1+\delta_{0j})\lambda_0 U(z) + (1+\mu)n^2\sigma^2\right) u_2 \end{pmatrix}, \end{aligned}$$

as well as the operator $M^{\delta,\rho}_{n,j,\mu}\colon C^{2,\delta}(\mathbb{R},\mathbb{R}^2)\to C^{0,\delta}(\mathbb{R},\mathbb{R}^2)$ with

$$\begin{aligned}M^{\delta,\rho}_{n,j,\mu}u &= e^{\rho z}L^{\delta,\rho}_{n,j,\mu}\left(ue^{-\rho z}\right)\\ &= \left(\begin{array}{c} u_1''+\left[\rho^2-\rho c_0-\left(\lambda_j-(1+\delta_{0j})\lambda_0U(z)+(1+\mu)n^2\sigma^2\right)\right]u_1-2n\sigma\left(u_2'-\rho u_2\right)\\ u_2''+\left[\rho^2-\rho c_0-\left(\lambda_j-(1+\delta_{0j})\lambda_0U(z)+(1+\mu)n^2\sigma^2\right)\right]u_2+2n\sigma\left(u_1'-\rho u_1\right)\end{array}\right).\end{aligned}$$

We may rewrite the above operator as

$$M^{\delta,\rho}_{n,j,\mu}u=\left(\begin{array}{cc}1&0\\0&1\end{array}\right)u''+\left(\begin{array}{cc}0&-2n\sigma\\2n\sigma&0\end{array}\right)u'+\left(\begin{array}{cc}q(z)&2n\sigma\rho\\-2n\sigma\rho&q(z)\end{array}\right)u=:I_2u''+\beta u'+\gamma(z)u,$$

with, given that $\rho=\frac{c_0}{2}$,

$$q(z):=-\frac{c_0^2}{4}-\left(\lambda_j-(1+\delta_{0j})\lambda_0U(z)+(1+\mu)n^2\sigma^2\right).$$

We also set

$$\gamma_\pm=\lim_{z\to\pm\infty}\gamma(z)=\left(\begin{array}{cc}q_\pm&n\sigma c_0\\-n\sigma c_0&q_\pm\end{array}\right),$$

$$\mathbb{R}\ni q_\pm=\lim_{z\to\pm\infty}q(z)=-\frac{c_0^2}{4}-\left(\lambda_j+(1+\mu)n^2\sigma^2\right)+\begin{cases}(1+\delta_{0j})\lambda_0, & \text{if } q_\pm=q_-,\\ 0, & \text{if } q_\pm=q_+.\end{cases}$$

Finally, we define

$$T^\pm_\rho\colon \xi\in\mathbb{R}\mapsto -\xi^2I+i\xi\beta+\gamma_\pm\in M_{2\times2}(\mathbb{R}).$$

In order to apply Theorem 34, we have to check if $T^\pm_\rho(\xi)$ is invertible for any $\xi\in\mathbb{R}$. We compute

$$\begin{aligned}\det T^\pm_\rho(\xi)&=\left|\begin{array}{cc}-\xi^2+q_\pm&-2in\sigma\xi+2n\sigma\rho\\2in\sigma\xi-2n\sigma\rho&-\xi^2+q_\pm\end{array}\right|\\&=\left(-\xi^2+q_\pm\right)^2+(2in\sigma\xi-2n\sigma\rho)^2\\&=P_{\mathbb{R}}(\xi)-8in^2\sigma^2\rho\xi,\end{aligned}$$

with $P_{\mathbb{R}}(\xi)$ a real polynomial function of ξ. Assume by contradiction that there exists $\xi\in\mathbb{R}$ such that $\det T^\pm_\rho(\xi)=0$. Then necessarily $n\xi=0$. Then n has to be zero, for otherwise we would have $\xi=0$, while

$$\det T^\pm_\rho(0)=q_\pm^2+(2n\sigma\rho)^2>0.$$

Therefore $n=0$, but this would yield

$$\det T^\pm_\rho(\xi)=\left(-\xi^2+q_\pm\right)^2.$$

However, since $(n,j)\neq(0,0)$

$$q_+=-\frac{c_0^2}{4}-\lambda_j<-\frac{1}{4}\left(c_0^2+4\lambda_0\right)\leq 0,\qquad q_-=q_++\lambda_0<q_+\leq 0,$$

which means that if $n=0$, we again have $\det T^\pm_\rho(\xi)\neq 0$ for any $\xi\in\mathbb{R}$, which contradicts our assumption. Consequently, $T^\pm_\rho(\xi)$ is invertible for any $\xi\in\mathbb{R}$.

Hence, from Theorem 34, we deduce that $L^\delta_{n,j,\mu}$ is Fredholm with

$$\operatorname{ind}M^{\delta,\rho}_{n,j,\mu}=k^+-k^-,\qquad k^\pm=\mathrm{Sp}M^\pm\cap\{z:\operatorname{Re}z>0\},$$

where

$$M^\pm:=\left(\begin{array}{cc}0&-I_2\\\gamma_\pm&\beta\end{array}\right)=\left(\begin{array}{cccc}0&0&-1&0\\0&0&0&-1\\q_\pm&n\sigma c_0&0&-2n\sigma\\-n\sigma c_0&q_\pm&2n\sigma&0\end{array}\right).$$

Let us first determine k^+. We compute

$$\begin{aligned}\det(M^+ - XI_4) &= \begin{vmatrix} q_+ + X^2 & n\sigma c_0 + 2n\sigma X \\ -n\sigma c_0 - 2n\sigma X & q_+ + X^2 \end{vmatrix} \\ &= (q_+ + X^2)^2 + (n\sigma c_0 + 2n\sigma X)^2 \\ &= \left(X^2 + q_+ - in\sigma c_0 - 2in\sigma X\right)\left(X^2 + q_+ + in\sigma c_0 + 2in\sigma X\right),\end{aligned}$$

from which we deduce that the eigenvalues of M^+ are

$$X^1_\pm = \frac{2in\sigma \pm \sqrt{(2in\sigma)^2 - 4(q_+ - in\sigma c_0)}}{2}, \quad X^2_\pm = \frac{-2in\sigma \pm \sqrt{(2in\sigma)^2 - 4(q_+ + in\sigma c_0)}}{2},$$

and thus

$$\begin{aligned}\mathrm{Re}\,(2X^1_\pm) &= \pm\,\mathrm{Re}\,\sqrt{-4n^2\sigma^2 - 4q_+ + 4in\sigma c_0} \\ &= \pm\,\mathrm{Re}\,\sqrt{c_0^2 + 4\lambda_j + 4in\sigma c_0 + 4\mu n^2\sigma^2} \\ &= \pm\,\mathrm{Re}\,(b^+_{n,j,\mu} - b^-_{n,j,\mu}).\end{aligned}$$

Notice that, since $(n,j) \neq (0,0)$, we have $\mathrm{Re}\,(b^+_{n,j,\mu} - b^-_{n,j,\mu}) > 0$, so that $\mathrm{Re}\,X^1_- < 0 < \mathrm{Re}\,X^1_+$. Similar calculations yield

$$\mathrm{Re}\,(2X^2_\pm) = \pm\,\mathrm{Re}\,\sqrt{c_0^2 + 4\lambda_j - 4in\sigma c_0 + 4\mu n^2\sigma^2} = \pm\,\mathrm{Re}\,(b^+_{-n,j,\mu} - b^-_{-n,j,\mu}),$$

therefore we also have $\mathrm{Re}\,X^2_- < 0 < \mathrm{Re}\,X^2_+$. As a result $k^+ = 2$.

Let us now turn our attention to k^-. Similar calculations yield that the eigenvalues of M^- are

$$Y^1_\pm = \frac{2in\sigma \pm \sqrt{(2in\sigma)^2 - 4(q_- - in\sigma c_0)}}{2}, \quad Y^2_\pm = \frac{-2in\sigma \pm \sqrt{(2in\sigma)^2 - 4(q_- + in\sigma c_0)}}{2},$$

which yields

$$\begin{aligned}\mathrm{Re}\,(2Y^1_\pm) &= \pm\,\mathrm{Re}\,\sqrt{-4n^2\sigma^2 + 4in\sigma c_0 - 4q_-} \\ &= \pm\,\mathrm{Re}\,\sqrt{c_0^2 + 4in\sigma c_0 + 4\lambda_j + 4\mu n^2\sigma^2 - 4(1+\delta_{0j})\lambda_0} \\ &= \pm\,\mathrm{Re}\,(a^+_{n,j,\mu} - a^-_{n,j,\mu}).\end{aligned}$$

Notice that $\mathrm{Re}\,(a^+_{n,j,\mu} - a^-_{n,j,\mu}) > 0$, so that $\mathrm{Re}\,Y^1_- < 0 < \mathrm{Re}\,Y^1_+$, and similarly $\mathrm{Re}\,Y^2_- < 0 < \mathrm{Re}\,Y^2_+$. Hence, $k^- = 2$, which means $\mathrm{ind}\,M^{\delta,\rho}_{n,j,\mu} = 0$. Since the operator $S_\rho\colon u \in C^{2,\delta}(\mathbb{R},\mathbb{R}^2) \mapsto ue^{-\rho z} \in C^{2,\delta}_\rho(\mathbb{R},\mathbb{R}^2)$ is continuously invertible with $S_\rho^{-1}\colon u \in C^{2,\delta}_\rho(\mathbb{R},\mathbb{R}^2) \mapsto ue^{\rho z} \in C^{2,\delta}(\mathbb{R},\mathbb{R}^2)$, we have that $L^{\delta,\rho}_{n,j,\mu}u = S_\rho M^{\delta,\rho}_{n,j,\mu} S_\rho^{-1} u$ shares the same Fredholm property and index as $M^{\delta,\rho}_{n,j,\mu}$. From there, we prove similarly that the operator $\mathcal{L}^{\delta,\rho}_{n,j,\mu}$ defined by (189) is also Fredholm and satisfies $\mathrm{ind}\,\mathcal{L}^{\delta,\rho}_{n,j,\mu} = 0$. The proof is thus complete. □

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