Lotka-Volterra competition-diffusion system: the critical competition case

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Abstract

We consider the reaction-diffusion competition system in the so-called *critical competition case*. The associated ODE system then admits infinitely many equilibria, which makes the analysis intricate. We first prove the non-existence of *ultimately monotone* traveling waves by applying the phase plane analysis. Next, we study the large time behavior of the solution of the Cauchy problem with a compactly supported initial datum. We not only reveal that the “faster” species excludes the “slower” one (with a known *spreading speed*), but also provide a sharp description of the profile of the solution, thus shedding light on a new *bump phenomenon*.

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1 Introduction

We consider the Lotka-Volterra competition-diffusion system

$$\begin{cases} \partial_t u = u_{xx} + u(1-u-v), & t>0, x\in\mathbb{R}, \\ \partial_t v = dv_{xx} + rv(1-v-u), & t>0, x\in\mathbb{R}, \end{cases} \tag{1.1}$$

which is *critical* among systems in the form of (1.5). The main difficulty is that the underlying ODE competition system

$$\begin{cases} u' = u(1-u-v), & t>0, \\ v' = rv(1-v-u), & t>0, \end{cases} \tag{1.2}$$

admits infinitely many (nontrivial) equilibria: the whole line $u+v=1$. Because of that, there are very few available mathematical results on system (1.1). In the present paper, we fill this gap by proving the non-existence of *ultimately monotone* traveling waves, and giving a very precise description of the large time behavior of the solution starting from a compactly supported initial datum, thus revealing a new *bump phenomenon*.

In the absence of one species, system (1.1) reduces to the reaction-diffusion equation

$$\partial_t u = du_{xx} + ru(1-u), \quad t>0, x\in\mathbb{R}, \tag{1.3}$$

introduced by Fisher [6] and Kolmogorov, Petrovsky and Piskunov [16] as a population genetics model to investigate the propagation of a dominant gene in a homogeneous environment. The KPP equation (1.3) has two main properties. Firstly, nonnegative traveling waves, corresponding to the ansatz $u(t,x) = U(x-ct)$ and solving

$$\begin{cases} dU'' + cU' + rU(1-U) = 0 & \text{in } \mathbb{R}, \\ U(-\infty) = 1, \ U(\infty) = 0, \end{cases} \tag{1.4}$$

exist if and only if their speeds $c \geq c^* := 2\sqrt{dr}$. Secondly, the solution of (1.3) starting from a nonnegative (nontrivial) compactly supported initial datum, satisfies

$$\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0, \qquad \text{for all } c > c^*,$$

$$\lim_{t\to\infty} \sup_{|x|\leq ct} |1-u(t,x)| = 0, \qquad \text{for all } c < c^*,$$

see [1]. In other words, the minimal speed c^* of traveling wave solutions corresponds to the *spreading speed* of the solution of the Cauchy problem with a compactly supported initial datum.

The general Lotka-Volterra competition-diffusion system is written

$$\begin{cases} \partial_t u = u_{xx} + u(1-u-av), & t>0, x\in\mathbb{R}, \\ \partial_t v = dv_{xx} + rv(1-v-bu), & t>0, x\in\mathbb{R}. \end{cases} \tag{1.5}$$

Here $u = u(t,x)$ and $v = v(t,x)$ represent the population densities of two competing species, $d > 0$ and $r > 0$ stand for the diffusion rate and intrinsic growth rate of v (while those of u have been normalized), and $a > 0$ and $b > 0$ represent the strength of v and u, respectively, as competitors. The parameters a and b determine the behavior of the underlying ODE system (see below) but, once fixed, the outcomes for system (1.5) are highly dependent on the parameters r, d and the initial datum. The situation is therefore very rich and we refer to the works mentioned below for more details and references.

The so-called *weak competition case* corresponds to $a < 1$ and $b < 1$. Nontrivial solutions of the underlying ODE system tend to the co-existence equilibrium. For the diffusion system, it was proved by Tang and Fife [24] that there exists a minimal speed $c^\star > 0$ such that a monotone traveling wave solution connecting the co-existence equilibrium to the null state $(0,0)$ exists if and only if $c \geq c^\star$, which is comparable to the Fisher-KPP equation mentioned above. Concerning the large time behavior of the Cauchy problem, some first estimates were obtained by Lin and Li [19]. More recently, Liu, Liu and Lam [20, 21] obtained some rather complete results.

The so-called *strong competition case* corresponds to $a > 1$ and $b > 1$. Since the co-existence equilibrium is unstable and the equilibria $(1,0)$ and $(0,1)$ are both stable for the underlying ODE system, this case corresponds to a *bistable* situation. For the diffusion system, it was proved by Kan-On [13], see also [7], that there exists a unique traveling wave solution connecting $(1,0)$ to $(0,1)$. The sign of the speed of this wave determines the "winner" between u and v, and thus is very relevant for applications, see the review [8]. We refer to [10], [11] and [23] for some results on this delicate issue. As far as the large time behavior of the Cauchy problem is concerned, we refer to the recent work of Carrère [3] revealing the possibility of *propagating terraces*, see [4, 5]. Very recently, Peng, Wu and Zhou [22] provided refined estimates of both the spreading speed and the profile of the solution.

Last, the so-called *strong-weak competition case* corresponds to $a < 1 < b$. Nontrivial solutions of the underlying ODE system tend to the state $(1,0)$ meaning that "u excludes v". For the diffusion system, the traveling wave solutions were constructed by Kan-On [14]. Among others, the important issue of the linear determinacy of the minimal traveling wave speed was first studied by Lewis, Li and Weinberger [17, 18]. Concerning the large time behavior of the solution of the Cauchy problem, Girardin and Lam [9] recently studied the spreading speed of solutions with an initial datum that is null (or exponentially decaying) on the right half line. They obtained a rather complete understanding of the spreading properties, revealing in particular the possibility of a so-called *nonlocal pulling phenomenon* (see Appendix of the present paper for more details).

In the present paper, our goal is to complete the above picture by considering the issues of both traveling wave solutions and the Cauchy problem in the so-called *critical competition case* $a = b = 1$, corresponding to system (1.1).

2 Main results

A traveling wave solution of system (1.1) is defined as follows.

Definition 2.1 ((α,β)-traveling wave)**.** *Let* $0 \leq \alpha, \beta \leq 1$ *be given with* $\alpha \neq \beta$*. Then an* (α,β)*-traveling wave solution (or traveling wave if there is no ambiguity) of* (1.1) *is a triplet*

(c, U, V)*, where* $c \in \mathbb{R}$ *is the traveling wave speed and* (U, V) *two nonnegative profiles, solving*

$$\begin{cases} U'' + cU' + U(1 - U - V) = 0, \\ dV'' + cV' + rV(1 - V - U) = 0, \\ (U, V)(-\infty) = (\alpha, 1 - \alpha), \\ (U, V)(+\infty) = (\beta, 1 - \beta). \end{cases} \tag{2.1}$$

As mentioned above, for both the strong competition case and the strong-weak competition case, *monotone* traveling waves connecting $(1, 0)$ to $(0, 1)$ are known to exist. In the critical competition case under consideration, our first main result is that there is no *ultimately monotone* traveling wave connecting any two different nonnegative steady states on the line $u + v = 1$.

Definition 2.2 (Ultimately monotone (α, β)-traveling wave)**.** *Let* $0 \leq \alpha, \beta \leq 1$ *be given with* $\alpha \neq \beta$*. Then an ultimately monotone* (α, β)*-traveling wave solution is an* (α, β)*-traveling wave for which there further exist* $-\infty < z_0 \leq z_0^* < +\infty$ *such that*

$$U'(z)V'(z) \neq 0, \quad \text{for all } z \in (-\infty, z_0] \cup [z_0^*, +\infty). \tag{2.2}$$

Remark 2.3. Obviously, if $(c, U(z), V(z))$ is an (ultimately monotone) (α, β)-traveling wave then $(-c, U(-z), V(-z))$ is a (ultimately monotone) (β, α)-traveling wave.

In other words, we do not require the traveling wave to be monotone on $\mathbb{R}$, but only to be monotone in some neighborhoods of both $-\infty$ and $+\infty$. This reinforces our non-existence result which states as follows.

Theorem 2.4 (Non-existence of ultimately monotone traveling waves)**.** *Let* $0 \leq \alpha, \beta \leq 1$ *be given with* $\alpha \neq \beta$*. Then, there is no ultimately monotone* (α, β)*-traveling wave for system* (1.1)*.*

The above theorem is proved in Section 3. The starting point consists in transforming system (2.1) into a first order system of four ODEs. Then, by a phase plane analysis, we prove that $U + V - 1$ has to "oscillate" in a neighborhood of $-\infty$ or $+\infty$, from which we get a contradiction.

Remark 2.5. As easily seen from the proof, to exclude the existence of a traveling wave with speed $c > 0$, $c < 0$, it is enough to assume that (2.2) holds in a neighborhood of $-\infty$, $+\infty$ respectively. In other words, there is no traveling wave for which the *invading state* is monotonically reached. The existence of a traveling wave for which the invading state is not monotonically reached remains an open issue. Last, as seen from subsection 3.1, the non existence of standing waves ($c = 0$) does not require any ultimately monotonicity assumption.

Our second main focus is concerned with the large time behavior of the solution of system (1.1) starting from a nonnegative (nontrivial) compactly supported initial datum. In both the strong competition case [3], [22], and the strong-weak competition case [9], the monotone traveling wave solutions of the entire system play a key role in studying the large time behavior of the solution of the Cauchy problem. However, for the critical competition case, such traveling wave solutions do not exist.

In order to state our result, we define the (minimal) Fisher-KPP traveling wave solution (c_u, U_{KPP}) as

$$c_u := 2, \quad \begin{cases} U_{KPP}'' + c_u U_{KPP}' + U_{KPP}(1 - U_{KPP}) = 0, \\ U_{KPP}(-\infty) = 1, \; U_{KPP}(\infty) = 0, \end{cases} \tag{2.3}$$

and, similarly, (c_v, V_{KPP}) as

$$c_v := 2\sqrt{dr}, \quad \begin{cases} dV''_{KPP} + c_v V'_{KPP} + rV_{KPP}(1 - V_{KPP}) = 0, \\ V_{KPP}(-\infty) = 1, \; V_{KPP}(\infty) = 0. \end{cases} \tag{2.4}$$

Let us recall that both U_{KPP} and V_{KPP} are uniquely defined "up to a shift". Note that, c_u (resp. c_v) also represents the spreading speed of u (resp. v) in the absence of v (resp. u).

Theorem 2.6 (Propagation phenomenon)**.** *Let $(u, v) = (u, v)(t, x)$ be the solution of system* (1.1) *starting from an initial datum $(u_0, v_0) = (u_0, v_0)(x)$ satisfying*

$$u_0 \text{ and } v_0 \text{ are continuous, nontrivial, compactly supported, and } 0 \le u_0, v_0 \le 1. \tag{2.5}$$

Then the following holds.

(i) Assume $dr > 1$ (i.e. $c_v > c_u$). Then

$$\lim_{t\to\infty} \left(\sup_{x\in\mathbb{R}} \left| v(t,x) - V_{KPP}\left(|x| - c_v t + \frac{3d}{c_v} \ln t + \eta_*(t) \right) \right| + \sup_{x\in\mathbb{R}} \; u(t,x) \right) = 0, \tag{2.6}$$

where η_* is a bounded function on $[0, \infty)$.

(ii) Assume $dr < 1$ (i.e. $c_v < c_u$). Then

$$\lim_{t\to\infty} \left(\sup_{x\in\mathbb{R}} \left| u(t,x) - U_{KPP}\left(|x| - c_u t + \frac{3}{c_u} \ln t + \eta_{**}(t) \right) \right| + \sup_{x\in\mathbb{R}} \; v(t,x) \right) = 0, \tag{2.7}$$

where η_{**} is a bounded function on $[0, \infty)$.

The above theorem is proved in Section 4. Let us briefly comment on Theorem 2.6. First of all, for the case $dr > 1$ (or $dr < 1$), the "faster species", namely v, excludes the "slower one", namely u, and imposes its spreading speed, see (2.6). Furthermore, we find that the profile of the solution uniformly converges to the corresponding minimal KPP traveling wave solution, and this up to an identified logarithmic Bramson correction, see (2.6) again.

On the other hand, for the case $dr = 1$, it is difficult to decide which species leads the invading front, and the behavior of the solution is highly depending on both the parameters and the shape of the initial datum. For the case $d = r = 1$, for any initial datum satisfying (2.5), a coexistence phenomenon happens. However, for the case $dr = 1$ but $d \neq 1$, the behavior of the solution is much more intricate. Indeed, in this case, the logarithmic phase drifts for u and v are different and there is a narrow region of width $O(\ln t)$ where the behaviors of u and v are difficult to "anticipate". This may cause some subtle phenomena (both species driving the front or one excluding the other) and makes the mathematical analysis quite involved. We hope to address these issues in a future work.

Our second result on the Cauchy problem deals with the region "$|x| \le \varepsilon_* t$", where the profile of the solution is more of the "Heat equation type".

Theorem 2.7 (Bump phenomenon)**.** *Let $(u, v) = (u, v)(t, x)$ be the solution of system* (1.1) *starting from an initial datum $(u_0, v_0) = (u_0, v_0)(x)$ satisfying* (2.5)*. Denote*

$$k^* := \min\left(\frac{1}{2d}, \frac{d}{2}\right), \quad d^* := \max(1, d).$$

Then the following holds.

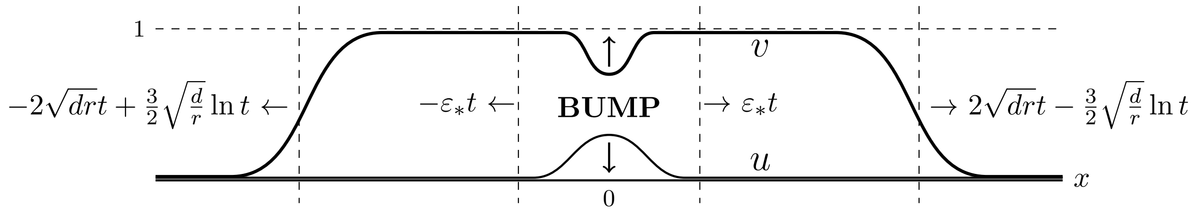

Figure 1: The asymptotic profile of the solution, in the case $dr > 1$.

(*i*) *Assume* $dr > 1$ *(i.e.* $c_v > c_u$*). Then for* $\varepsilon_* > 0$ *small enough and* $0 < \theta < \frac{1}{2}$*, there exist* $C_2 > C_1 > 0$ *and* $T > 0$ *such that both*

$$C_1 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \leq u(t,x) \leq C_2 t^{-k^*} e^{-\frac{x^2}{4d^* t}}, \tag{2.8}$$

$$\max\left(C_1 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} - t^{-(1+\theta)}, 0\right) \leq 1 - v(t,x) \leq C_2 t^{-k^*} e^{-\frac{x^2}{4d^* t}}, \tag{2.9}$$

hold for any $t \geq T$*,* $|x| \leq \varepsilon_* t$*.*

(*ii*) *Assume* $dr < 1$ *(i.e.* $c_v < c_u$*). Then for* $\varepsilon_{**} > 0$ *small enough and* $0 < \theta < \frac{1}{2}$*, there exist* $C_4 > C_3 > 0$ *and* $T > 0$ *such that both*

$$C_3 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \leq v(t,x) \leq C_4 t^{-k^*} e^{-\frac{x^2}{4d^* t}}, \tag{2.10}$$

$$\max\left(C_3 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} - t^{-(1+\theta)}, 0\right) \leq 1 - u(t,x) \leq C_4 t^{-k^*} e^{-\frac{x^2}{4d^* t}}, \tag{2.11}$$

hold for any $t \geq T$*,* $|x| \leq \varepsilon_{**} t$*.*

The above theorem is proved in Section 5. Let us briefly comment on Theorem 2.7, say in the case $dr > 1$. As revealed by (2.8) and (2.9), the solution converges to $(0,1)$ *exponentially* in regions of the form $|x| \geq \varepsilon t$ with $\varepsilon > 0$, but only *algebraically* in "sublinear regions" of the form $|x| \lesssim \sqrt{t}$. We call this a *bump phenomenon*, see Figure 1. Such a phenomenon does not occur in the strong competition case [22]. As far as the strong-weak competition case is concerned, the results as stated in [9] are not sufficient to decide if it occurs or not, but we assert it does not, as proved in the forthcoming work [25]. Therefore, the present paper is the first one revealing a bump phenomenon in the context of competition-diffusion systems. We believe such a phenomenon is reserved for the critical case $a = b = 1$, and is rare to happen in the context of homogeneous reaction-diffusion equations.

3 Non-existence of traveling waves

This section is devoted to the proof of Theorem 2.4 on the non-existence of ultimately monotone traveling waves for system (1.1).

3.1 Preliminary results and observations

In this subsection, the ultimately monotonicity assumption is not required, and thus a traveling wave is understood in the sense of Definition 2.1. We start with the following *a priori* estimates for a traveling wave.

Lemma 3.1. *Any traveling wave has to satisfy* $0 < U < 1$, $0 < V < 1$, *and* $U'(\pm\infty) = V'(\pm\infty) = 0$.

Proof. The positivity of the profiles follows from the strong maximum principle. If $U \le 1$ is not true, then U has to reach a maximum value strictly larger than 1 at some point, and evaluating the U-equation at this point gives a contradiction. Hence $U \le 1$ and, from the strong maximum principle, $U < 1$. Similarly, one has $V < 1$.

We now prove the limit behavior $U'(+\infty) = 0$, the other ones being proved similarly. Denote the set of accumulation points of U' in $+\infty$ by $\mathcal{A}$. Since U is bounded, $0 \in \mathcal{A}$. Let $\ell \in \mathcal{A}$. Then there exists a sequence $z_n \to +\infty$ such that $U'(z_n) \to \ell$ as $n \to +\infty$. Then $(U_n, V_n)(z) := (U, V)(z + z_n)$ solves

$$U_n'' + cU_n' = -U_n(1 - U_n - V_n).$$

Since the L^∞ norm of the right hand side term is uniformly bounded with respect to n, the interior elliptic estimates imply that, for all $R > 0$ and $1 < p < \infty$, the sequence (U_n) is bounded in $W^{2,p}(-R, R)$. From Sobolev embedding theorem we have that, up to a subsequence, U_n converges to some U_∞ in $C^1_{loc}(\mathbb{R})$. The boundary condition $U(+\infty) = \beta$ thus enforces $U_\infty \equiv \beta$ and $U_\infty' \equiv 0$. As a result, $U'(z_n) = U_n'(0) \to U_\infty'(0) = 0$, and thus $\ell = 0$. Hence $\mathcal{A} = \{0\}$, which concludes the proof. □

We conclude this subsection by showing the non-existence of traveling wave solutions for two special cases.

Proposition 3.2 (Non-existence of standing waves)**.** *There is no standing wave, i.e. traveling wave with speed* $c = 0$, *for system* (1.1).

Proof. Assume $c = 0$. By adding the both sides of the U-equation and the V-equation, we find that $W := U + V$ satisfies

$$W'' + \left(U + \frac{r}{d}V\right)(1 - W) = 0, \quad W(\pm\infty) = 1.$$

If $W \not\equiv 1$, then W reaches either a maximum value strictly larger than 1 or a minimum value in $(0, 1)$, which is impossible from the above equation (recall that $U + \frac{r}{d}V > 0$). As a result $W = U + V \equiv 1$. Going back to the original equations we have $U'' = V'' = 0$. Since U and V are bounded, U and V must be constant, which cannot happen since $\alpha \neq \beta$. □

Proposition 3.3 (Non-existence of traveling waves when $d = 1$)**.** *Assume* $d = 1$. *Then there is no traveling wave for system* (1.1).

Proof. Again, by adding the both sides of the U-equation and V-equation, we see that $W := U + V$ satisfies

$$W'' + cW' + (U + rV)(1 - W) = 0, \quad W(\pm\infty) = 1,$$

so that, as in Proposition 3.2, we have $W \equiv 1$, and thus $U'' + cU' = 0$. From Lemma 3.1, by integrating both sides from $-\infty$ to $+\infty$, we have $c(\beta - \alpha) = 0$, which yields a contradiction since $c \neq 0$ and $\alpha \neq \beta$. □

3.2 Proof of Theorem 2.4

We now consider the case of ultimately monotone traveling waves.

Lemma 3.4. *Let $0 \leq \alpha, \beta \leq 1$ be given with $\alpha \neq \beta$. Let (c, U, V) be an ultimately monotone (α, β)-traveling wave solution. Then* (2.2) *is refined in*

$$U'(z)V'(z) < 0, \quad \textit{for all } z \in (-\infty, z_0] \cup [z_0^*, +\infty). \tag{3.1}$$

Proof. We only deal with the behavior around $-\infty$. If the conclusion is false, we may assume that $U'(z) > 0$ and $V'(z) > 0$ for all $z \in (-\infty, z_0]$, the case $U'(z) < 0$ and $V'(z) < 0$ being treated similarly. From the boundary conditions $(U, V)(-\infty) = (\alpha, 1-\alpha)$ and $(U, V)(+\infty) = (\beta, 1-\beta)$, there must exist a point $z_1 > z_0$ such that

$$U'(z) > 0, V'(z) > 0, \quad \text{for all } z < z_1, \quad U'(z_1) = 0 \text{ or } V'(z_1) = 0.$$

Assume w.l.o.g. that $U'(z_1) = 0$. In particular $U''(z_1) \leq 0$. From the U-equation, this enforces $(U + V)(z_1) \leq 1$ which contradicts to $(U + V)(-\infty) = 1$ and $(U + V)' > 0$ on $(-\infty, z_1)$. □

We now prove, in the case $d \neq 1$, the non-existence of ultimately monotone traveling waves with speed $c \neq 0$. In view of subsection 3.1, this is enough to complete the proof of Theorem 2.4.

Completion of the proof of Theorem 2.4. For $d \neq 1$, let us consider (c, U, V) an ultimately monotone (α, β)-traveling wave with $c \neq 0$. In the sequel we only deal with the case $c > 0$ for which we perform a phase plane analysis around $-\infty$ (for the case $c < 0$, one has to perform a phase plane analysis around $+\infty$ with similar arguments). We define $W := \alpha - U$, $P := U'$, $R := V - 1 + \alpha$, $Q := V'$. Then we can rewrite (2.1) as

$$\begin{cases} W' = -P, \\ P' = -cP - (\alpha - W)(W - R), \\ R' = Q, \\ Q' = -\dfrac{c}{d}Q - \dfrac{r}{d}(R + 1 - \alpha)(W - R), \\ (W, P, R, Q)(-\infty) = (0, 0, 0, 0), \\ (W, P, R, Q)(+\infty) = (\alpha - \beta, 0, \alpha - \beta, 0). \end{cases} \tag{3.2}$$

Assume that $W - R$ is ultimately nonnegative, that is

$$\exists z^* < z_0, \forall z \leq z^*, (W - R)(z) \geq 0. \tag{3.3}$$

From (3.1), we know that it holds either $P > 0$ or $Q > 0$ on $(-\infty, z_0]$. Moreover, from Lemma 3.1, $\alpha - W \geq 0$ and $R + 1 - \alpha \geq 0$. If $P > 0$ on $(-\infty, z_0]$, then from the P-equation in (3.2), we have $P' < 0$ on $(-\infty, z^*]$, which contradicts to $P > 0$ on $(-\infty, z_0]$ and $P(-\infty) = 0$. If $Q > 0$ on $(-\infty, z_0]$, we similarly get a contradiction from the Q-equation. Hence (3.3) does not hold.

Assume that $W - R$ is ultimately nonpositive, that is

$$\exists z^* < z_0, \forall z \leq z^*, (W - R)(z) \leq 0. \tag{3.4}$$

From (3.1) again, we know that it holds either $P < 0$ or $Q < 0$ on $(-\infty, z_0]$. If $P < 0$ on $(-\infty, z_0]$, then from the P-equation in (3.2), we have $P' > 0$ on $(-\infty, z^*]$, which contradicts to $P < 0$ on $(-\infty, z_0]$ and $P(-\infty) = 0$. If $Q < 0$ on $(-\infty, z_0]$, we similarly get a contradiction from the Q-equation. Hence (3.4) does not hold.

As a result, since $W - R$ is neither ultimately nonnegative nor ultimately nonpositive, we can find a local maximum point $z_1 < z_0$ and a local minimum point $z_2 < z_0$ such that

$$(W - R)(z_1) > 0,\ (W - R)'(z_1) = 0,\ (W - R)''(z_1) \leq 0;$$

$$(W - R)(z_2) < 0,\ (W - R)'(z_2) = 0,\ (W - R)''(z_2) \geq 0.$$

Note that

$$(W - R)'' = cP + \frac{c}{d}Q + \left(\alpha - W + \frac{r}{d}(R + 1 - \alpha)\right)(W - R).$$

From (3.1), it holds either $Q > 0$ or $Q < 0$ on $(-\infty, z_0]$. Let us first consider the case $d < 1$. If $Q > 0$ on $(-\infty, z_0]$, since $(W-R)'(z_1) = 0$ means $(P+Q)(z_1) = 0$, we have $(cP+\frac{c}{d}Q)(z_1) > 0$. Therefore, from the above equation, $(W-R)''(z_1) > 0$, which is a contradiction. On the other hand, if $Q < 0$ on $(-\infty, z_0]$, since $(P + Q)(z_2) = 0$, we have $(cP + \frac{c}{d}Q)(z_2) < 0$. Therefore, from the above equation, $(W - R)''(z_2) < 0$, which is a contradiction. Last, the case $d > 1$ can be treated similarly.

Therefore, we conclude that system (1.1) does not admit any ultimately monotone traveling wave. □

4 The Cauchy problem

In this section, we consider system (1.1) with a compactly supported initial datum, and prove the propagation result, namely Theorem 2.6.

4.1 Preliminaries

Let us start by briefly recalling the competitive comparison principle. Define the operators

$$N_1[u, v] := u_t - u_{xx} - u(1 - u - v) \quad \text{and} \quad N_2[u, v] := v_t - dv_{xx} - rv(1 - v - u).$$

Consider a domain $\Omega := (t_1, t_2) \times (x_1, x_2)$ with $0 \leq t_1 < t_2 \leq +\infty$ and $-\infty \leq x_1 < x_2 \leq +\infty$. A (classical) super-solution is a pair $(\overline{u}, \underline{v}) \in \left[C^1\left((t_1, t_2), C^2((x_1, x_2))\right) \cap C_b\left(\overline{\Omega}\right)\right]^2$ satisfying

$$N_1[\overline{u}, \underline{v}] \geq 0 \quad \text{and} \quad N_2[\overline{u}, \underline{v}] \leq 0 \ \text{ in } \Omega.$$

Similarly, a (classical) sub-solution $(\underline{u}, \overline{v})$ requires $N_1[\underline{u}, \overline{v}] \leq 0$ and $N_2[\underline{u}, \overline{v}] \geq 0$.

Proposition 4.1 (Comparison Principle)**.** *Let $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$ be a super-solution and sub-solution of system (1.1) in Ω, respectively. If*

$$\overline{u}(t_1, x) \geq \underline{u}(t_1, x) \quad \text{and} \quad \underline{v}(t_1, x) \leq \overline{v}(t_1, x), \quad \text{for all } x \in (x_1, x_2),$$

and, for $i = 1, 2$,

$$\overline{u}(t, x_i) \geq \underline{u}(t, x_i) \quad \text{and} \quad \underline{v}(t, x_i) \leq \overline{v}(t, x_i), \quad \text{for all } t \in (t_1, t_2),$$

then, it holds

$$\overline{u}(t,x) \geq \underline{u}(t,x) \quad \textit{and} \quad \underline{v}(t,x) \leq \overline{v}(t,x), \quad \textit{for all } (t,x) \in \Omega.$$

If $x_1 = -\infty$ or $x_2 = +\infty$, the hypothesis on the corresponding boundary condition can be omitted.

Denote $(u,v) = (u,v)(t,x)$ as the solution of (1.1) starting from $(u_0,v_0) = (u_0,v_0)(x)$ satisfying (2.5). Obviously, $(1,0)$ is a super-solution while $(0,1)$ is a sub-solution. It thus follows from (2.5), the comparison principle and the strong maximum principle that

$$0 < u(t,x) < 1 \text{ and } 0 < v(t,x) < 1, \quad \text{for all } \ t > 0, x \in \mathbb{R}. \tag{4.1}$$

Actually, the comparison principle also holds for the so-called *generalized* sub- and super-solutions. This is a rather well-known fact, and we refer to the clear exposition in [9, subsection 2.1] for more details. In particular, if $(\underline{u}_1, \overline{v})$ and $(\underline{u}_2, \overline{v})$ are both classical sub-solutions, then $(\max(\underline{u}_1, \underline{u}_2), \overline{v})$ is a generalized sub-solution. Also, if $(\underline{u}, \overline{v}_1)$ and $(\underline{u}, \overline{v}_2)$ are both classical sub-solutions, then $(\underline{u}, \min(\overline{v}_1, \overline{v}_2))$ is a generalized sub-solution.

We now start the proof of Theorem 2.6. Observe that, by changing the variables as $x = \sqrt{\frac{d}{r}}y$ and $t = \frac{1}{r}s$, system (1.1) can be rewritten as

$$\begin{cases} \partial_s v = v_{yy} + v(1-v-u), & s > 0, y \in \mathbb{R}, \\ \partial_s u = d^{-1}u_{xx} + r^{-1}u(1-u-v), & s > 0, y \in \mathbb{R}. \end{cases}$$

Therefore, without loss of generality, we assume from now that $dr > 1$, that is $c_v > c_u$, and shall prove the statement (i) in Theorem 2.6.

Since $c_v > c_u$ and u cannot propagate faster than c_u, the behavior of the solution in the region $|x| > c_u t$ is rather well-understood.

Proposition 4.2 (Estimates in the region $|x| > c_u t$)**.** *We have*

$$\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0, \quad \textit{for all } c > c_u, \tag{4.2}$$

$$\lim_{t\to\infty} \sup_{|x|\geq ct} v(t,x) = 0, \quad \textit{for all } c > c_v, \tag{4.3}$$

and

$$\lim_{t\to\infty} \sup_{c_1 t\leq |x|\leq c_2 t} |1 - v(t,x)| = 0, \quad \textit{for all } c_u < c_1 < c_2 < c_v. \tag{4.4}$$

Proof. Without loss of generality, we only deal with the case $x \geq 0$. Define

$$U(t,x) := C_1 e^{-\frac{c_u}{2}(x - c_u t)} \quad \text{and} \quad V(t,x) := C_2 e^{-\frac{c_v}{2d}(x - c_v t)},$$

where $C_1 > 0$ and $C_2 > 0$ are chosen large enough so that $U(0,\cdot) \geq u_0$ and $V(0,\cdot) \geq v_0$. We can easily check that $(U,0)$ is a super-solution while $(0,V)$ is a sub-solution. As a result, we have

$$0 < u(t,x) \leq \min\left(1, C_1 e^{-\frac{c_u}{2}(x-c_u t)}\right) \quad \text{and} \quad 0 < v(t,x) \leq \min\left(1, C_2 e^{-\frac{c_v}{2d}(x - c_v t)}\right), \tag{4.5}$$

from which (4.2) and (4.3) follow.

Next, let $c_u < c_1 < c_2 < c_v$ be given. Select $0 < a < 1 < b$ and consider (u^*, v^*) the solution of the strong-weak competition system

$$\begin{cases} \partial_t u^* = u^*_{xx} + u^*(1 - u^* - av^*), \\ \partial_t v^* = dv^*_{xx} + rv^*(1 - v^* - bu^*), \end{cases} \tag{4.6}$$

starting from (u_0, v_0). Obviously, (u^*, v^*) is a super-solution for system (1.1), and thus $v^*(t,x) \leq v(t,x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$. Since the statements (2) and (3) in [9, Theorem 1.1] imply

$$\lim_{t\to\infty} \sup_{c_1 t \leq x \leq c_2 t} |1 - v^*(t,x)| = 0,$$

the same conclusion holds for v. □

4.2 Construction of the super-solution

The goal of this subsection is to construct an adequate super-solution in, roughly speaking, the region $|x| < c_u t$. More precisely, let $\frac{1}{d} < r_1 < r$ be given and define $c_v^* := 2\sqrt{dr_1} < c_v$. In the sequel, we introduce V_1 as a traveling wave solution with speed $c_v^* = 2\sqrt{dr_1}$ solving

$$\begin{cases} dV_1'' + c_v^* V_1' + r_1 V_1(1 - V_1) = 0, \\ V_1(-\infty) = 1, \ V_1(\infty) = 0. \end{cases} \tag{4.7}$$

As well-known, $V_1' < 0$ and there are $\lambda_1 > 0$ and $M_1 > 0$ such that

$$1 - V_1(\xi) \sim M_1 e^{\lambda_1 \xi} \quad \text{as } \xi \to -\infty. \tag{4.8}$$

Let us fix $c_u < c_1 < c_v^*$. For $T > 0$, we will work in the domain (which is "expanding in time")

$$\Omega_1(T) := \{(t,x) \in (T,\infty) \times \mathbb{R} : |x| < c_1 t\}. \tag{4.9}$$

It turns out that the construction of the super-solution is highly dependent on the value of d.

• **The case $d \leq 1$.** We introduce $s = s(t,x)$ as the solution of the Cauchy problem

$$\begin{cases} \partial_t s = s_{xx}, \\ s(0,x) = s_0(x) := B_1 e^{-q|x|}, \end{cases} \tag{4.10}$$

and look for a super-solution $(\tilde{U}, \tilde{V})$ in the form

$$\begin{cases} \tilde{U}(t,x) := t^{\frac{1-d}{2}}(1 - e^{-\tau t}) s(t,x), \\ \tilde{V}(t,x) := V_1(x - c_v^* t) + V_1(-x - c_v^* t) - 1 - \tilde{U}(t,x). \end{cases} \tag{4.11}$$

All parameters that will be determined below (namely B_1, q and τ) are positive, and $q < 1$.

• **The case $d \geq 1$.** We introduce $s = s(t,x)$ as the solution of the Cauchy problem

$$\begin{cases} \partial_t s = ds_{xx}, \\ s(0,x) = s_0(x) := B_1 e^{-q|x|}, \end{cases} \tag{4.12}$$

and look for a super-solution $(\tilde{U}, \tilde{V})$ in the form

$$\begin{cases} \tilde{U}(t,x) := t^{\frac{d-1}{2d}}(1-e^{-\tau t})s(t,x), \\ \tilde{V}(t,x) := V_1(x-c_v^*t) + V_1(-x-c_v^*t) - 1 - \tilde{U}(t,x). \end{cases} \tag{4.13}$$

All parameters that will be determined below (namely B_1, q and τ) are positive, and $q < \frac{1}{d}$.

Obviously, (4.10)—(4.11) and (4.12)—(4.13) coincide when $d = 1$.

Proposition 4.3 (Super-solutions)**.** *The following holds.*

(i) Assume $d \leq 1$. Let $0 < q < 1$ and $0 < \tau < \lambda_1(c_v^* - c_1)$ be given. Then there exists $T^* > 0$ such that, for all $B_1 > 0$, $(\tilde{U}, \tilde{V})$, given by (4.10)—(4.11)*, is a super-solution in the domain $\Omega_1(T^*)$ as defined in* (4.9)*.*

(ii) Assume $d \geq 1$. Let $0 < q < \frac{1}{d}$ and $0 < \tau < \lambda_1(c_v^* - c_1)$ be given. Then there exists $T^* > 0$ such that, for all $B_1 > 0$, $(\tilde{U}, \tilde{V})$, given by (4.12)—(4.13)*, is a super-solution in the domain $\Omega_1(T^*)$ as defined in* (4.9)*.*

Proof. Since our super-solutions are even functions, it is enough to deal with $x \geq 0$. In other words, we work for $t \geq T$ (with $T > 0$ to be selected) and $0 \leq x < c_1 t$, with $c_u < c_1 < c_v^*$. For ease of notations, we shall use the shortcuts $\xi_\pm := \pm x - c_v^* t$. Since $\xi_- \leq -c_v^* t$ and $\xi_+ \leq -(c_v^* - c_1)t$, it follows from $V_1' < 0$ and (4.8) that there exist $C_- > 0$ and $C_+ > 0$ such that, for $T > 0$ large enough,

$$1 - V_1(\xi_-) \leq C_- e^{-\lambda_1 c_v^* t} \quad \text{and} \quad 1 - V_1(\xi_+) \leq C_+ e^{-\lambda_1(c_v^* - c_1)t}, \quad \text{for all } (t,x) \in \Omega_1^+(T), \tag{4.14}$$

where $\Omega_1^+(T) := \Omega_1(T) \cap (T, \infty) \times [0, \infty)$. Moreover, up to enlarging $T > 0$ if necessary, there exists $0 < \rho < \frac{1}{3}$ such that

$$0 < 1 - V_1(\xi_\pm) \leq \rho, \quad \text{for all } (t,x) \in \Omega_1^+(T). \tag{4.15}$$

We first assume $d \leq 1$. Some straightforward computations combined with (4.10) yield

$$N_1[\tilde{U}, \tilde{V}] = t^{\frac{1-d}{2}}(1-e^{-\tau t})s\left(\frac{1-d}{2}t^{-1} + \frac{\tau e^{-\tau t}}{1-e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-)\right).$$

In view of (4.14), by choosing $\tau < \lambda_1(c_v^* - c_1)$, we deduce that, for $T > 0$ large enough, $N_1[\tilde{U}, \tilde{V}] \geq 0$ in $\Omega_1^+(T)$. On the other hand, some straightforward computations combined with (4.10) and (4.7) yield

$$N_2[\tilde{U}, \tilde{V}] = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &:= t^{\frac{1-d}{2}}(1-e^{-\tau t})s\left(r\Big(2 - V_1(\xi_+) - V_1(\xi_-) - \frac{\tau e^{-\gamma t}}{r(1-e^{-\tau t})}\Big) - \frac{1-d}{2}t^{-1} - (1-d)\frac{\partial_t s}{s}\right), \\ J_2 &:= (r_1 - r)V_1(\xi_+)(1 - V_1(\xi_+)), \\ J_3 &:= (1 - V_1(\xi_-))\left((r_1 - r)V_1(\xi_-) + r(2 - 2V_1(\xi_+))\right). \end{aligned}$$

Since $r_1 < r$ and $0 < V_1 < 1$, we have $J_2 \leq 0$. Next, from (4.15), we have

$$J_3 \leq (1 - V_1(\xi_-))\left((r_1 - r)(1-\rho) + r(2 - 2V_1(\xi_+))\right),$$

which, in view of (4.14), is nonpositive up to enlarging $T > 0$ if necessary. Last, from the "Heat kernel expression" of $s(t,x)$, namely

$$s(t,x) = \left(G(t,\cdot) * s_0\right)(x), \quad \text{where } G(t,x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}},$$

we can check that $\partial_t s(t,x) \geq -\frac{1}{2t} s(t,x)$. As a result, since $d \leq 1$, we have

$$J_1 \leq t^{\frac{1-d}{2}} (1 - e^{-\tau t}) sr \left(2 - V_1(\xi_+) - V_1(\xi_-) - \frac{\tau e^{-\tau t}}{r(1 - e^{-\tau t})} \right).$$

In view of (4.14) and $\tau < \lambda_1(c_v^* - c_1)$, we have $J_1 \leq 0$ up to enlarging $T > 0$ if necessary. As a result, $N_2[\tilde{U}, \tilde{V}] \leq 0$ in $\Omega_1^+(T)$.

Next, we assume $d \geq 1$. Some straightforward computations combined with (4.12) yield

$$N_1[\tilde{U}, \tilde{V}] = t^{\frac{d-1}{2d}} (1 - e^{-\tau t}) s \left(\frac{d-1}{2d} t^{-1} + \frac{d-1}{d} \frac{\partial_t s}{s} + \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-) \right).$$

As above, since $d \geq 1$, $\partial_t s(t,x) \geq -\frac{1}{2t} s(t,x)$ implies

$$N_1[\tilde{U}, \tilde{V}] \geq t^{\frac{d-1}{2d}} (1 - e^{-\tau t}) s \left(\frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - 2 + V_1(\xi_+) + V_1(\xi_-) \right).$$

In view of (4.14) and $\tau < \lambda_1(c_v^* - c_1)$, we deduce that, for $T > 0$ large enough, $N_1[\tilde{U}, \tilde{V}] \geq 0$ in $\Omega_1^+(T)$. On the other hand, some straightforward computations combined with (4.12) and (4.7) yield

$$N_2[\tilde{U}, \tilde{V}] = J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 &:= t^{\frac{d-1}{2d}} (1 - e^{-\tau t}) s \left(r(2 - V_1(\xi_+) - V_1(\xi_-)) - \frac{\tau e^{-\tau t}}{1 - e^{-\tau t}} - \frac{d-1}{2d} t^{-1} \right), \\
J_2 &:= (r_1 - r) V_1(\xi_+)(1 - V_1(\xi_+)), \\
J_3 &:= (1 - V_1(\xi_-)) \left((r_1 - r) V_1(\xi_-) + r(2 - 2V_1(\xi_+)) \right).
\end{aligned}$$

By applying the same argument as that for $d \leq 1$, we get $N_2[\tilde{U}, \tilde{V}] \leq 0$ in $\Omega_1^+(T)$. □

Note that, time T^* in Proposition 4.3 is independent on $B_1 > 0$, which leaves "some room" to enlarge B_1 so that the "initial order" and the "order on the boundary of the domain" are suitable for the comparison principle to be applicable.

Proposition 4.4 (First estimate on (u,v))**.** *There exist $0 < q < \min(1, \frac{1}{d})$, $T^{**} > 0$ and $B_1 > 0$ such that*

$$u(t,x) \leq \tilde{U}(t,x) \quad \text{and} \quad \tilde{V}(t,x) \leq v(t,x), \quad \text{for all } t \geq T^{**}, |x| \leq c_1 t,$$

where $(\tilde{U}, \tilde{V})$ is given by (4.10)—(4.11) when $d \leq 1$, and by (4.12)—(4.13) when $d \geq 1$.

Proof. We aim at applying the comparison principle in $\Omega_1(T)$, as defined in (4.9), with a well-chosen $T>0$. Select $0<q<\min(1,\frac{1}{d})$ small enough so that

$$\max(qc_1-q^2, qc_1-dq^2)<c_1-c_u. \tag{4.16}$$

From Proposition 4.3, for any $T\geq T^*$, we are equipped with a super-solution $(\tilde U,\tilde V)$ for which $B_1>0$ is arbitrary. We only deal with the case $d\leq 1$, the case $d\geq 1$ being similar.

We first focus on $x=c_1t$, $t\geq T^*$ (the case $x=-c_1t$, $t\geq T^*$ being similar). Let us prove that, up to enlarging $T^*>0$ if necessary, it holds

$$u(t,c_1t)\leq \tilde U(t,c_1t), \quad \text{for all } t\geq T^*. \tag{4.17}$$

From the proof of Proposition 4.2, we know that $u(t,c_1t)\leq C_1e^{-(c_1-c_u)t}$ (recall that $c_u=2$). Recalling $s_0(x)=B_1e^{-q|x|}$, we have

$$s(t,x)=\frac{B_1}{\sqrt{4\pi t}}\left(\int_{-\infty}^0 e^{-\frac{(x-y)^2}{4t}}e^{qy}dy+\int_0^{+\infty}e^{-\frac{(x-y)^2}{4t}}e^{-qy}dy\right),$$

which can be recast, after some elementary computations,

$$s(t,x)=\frac{B_1}{\sqrt{\pi}}\left(e^{q^2t-qx}\int_{\frac{2qt-x}{2\sqrt{t}}}^{+\infty}e^{-w^2}dw+e^{q^2t+qx}\int_{\frac{2qt+x}{2\sqrt{t}}}^{+\infty}e^{-w^2}dw\right). \tag{4.18}$$

In particular, since $2q<2<c_1$, we have, by enlarging $T^*>0$ if necessary,

$$\tilde U(t,c_1t)\geq \frac{1}{2}s(t,c_1t)\geq \frac{B_1}{4}e^{-(qc_1-q^2)t}\geq \frac{B_1}{4}e^{-(c_1-c_u)t}.$$

The last inequality holds from the choice (4.16). Thus $B_1>4C_1$ is enough to get (4.17).

Let us recall that $v\geq v^*$ where (u^*,v^*) is the solution of the strong-weak competition system (4.6) with the same initial datum (u_0,v_0). From Lemma A.1 *(ii)* (see Appendix), up to enlarging T^* if necessary, there exist $\mu>0$ and $K>0$ such that $v^*(t,c_1t)\geq 1-Ke^{-\mu t}$ for all $t\geq T^*$. On the other hand, the construction of $\tilde V$ implies that $\tilde V(t,c_1t)\leq 1-\tilde U(t,c_1t)$. Therefore, by choosing $qc_1-q^2<\mu$, up to enlarging $T^*>0$ if necessary, we have

$$\tilde V(t,c_1t)\leq v(t,c_1t), \quad \text{for all } t\geq T^*. \tag{4.19}$$

Now, $q>0$ and $T^*>0$ are fixed from the above discussion. We focus on the initial datum, namely $t=T^*$, $|x|\leq c_1T^*$. As above, we deduce from (4.18) that

$$\inf_{|x|\leq c_1T^*}\tilde U(T^*,x)\geq \frac{1}{2}s(T^*,c_1T^*)\geq \frac{B_1}{4}e^{-(qc_1-q^2)T^*}\geq 1\geq \sup_{t>0,x\in\mathbb{R}}u(t,x),$$

provided that $B_1>0$ is large enough. On the other hand,

$$\sup_{|x|\leq c_1T^*}\tilde V(T^*,x)\leq 1-\inf_{|x|\leq c_1T^*}\tilde U(T^*,x)\leq 1-\frac{B_1}{4}e^{-(qc_1-q^2)T^*}\leq 0\leq \inf_{t>0,x\in\mathbb{R}}v(t,x),$$

provided that $B_1>0$ is large enough.

As a consequence, the comparison principle can be applied in $\Omega_1(T^*)$, which concludes the proof of Proposition 4.4. □

4.3 Proof of Theorem 2.6

From the discussion above, we are now in the position to obtain the following spreading speed result.

Proposition 4.5 (Spreading speed)**.** *Let* $(u, v) = (u, v)(t, x)$ *be the solution of* (1.1) *starting from an initial datum* $(u_0, v_0) = (u_0, v_0)(x)$ *satisfying (2.5). Then the following holds.*

(*i*) *Assume* $dr > 1$ *(i.e.* $c_v > c_u$*). Then, for any* $0 < c_1 < c_v < c_2$*,*

$$\lim_{t\to\infty}\left(\sup_{x\in\mathbb{R}} u(t,x) + \sup_{|x|\le c_1 t}|1-v(t,x)| + \sup_{|x|\ge c_2 t} v(t,x)\right) = 0. \tag{4.20}$$

(*ii*) *Assume* $dr < 1$ *(i.e.* $c_v < c_u$*). Then, for any* $0 < c_3 < c_u < c_4$*,*

$$\lim_{t\to\infty}\left(\sup_{x\in\mathbb{R}} v(t,x) + \sup_{|x|\le c_3 t}|1-u(t,x)| + \sup_{|x|\ge c_4 t} u(t,x)\right) = 0. \tag{4.21}$$

Proof. Let us prove (i). The result on u in (4.20) is obtained by combining (4.2) and Proposition 4.4. Next, for a given $0 < c_1 < c_v$, we select $c_1 < c_v^* < c_v$. Then, Proposition 4.4 yields $\sup_{|x|\le c_1 t}|1-v(t,x)| \to 0$ as $t\to\infty$. The last part of (4.20) is nothing else than the estimate (4.3). □

We are now in the position to complete the proof of Theorem 2.6.

Proof of Theorem 2.6 (i). Since the proof for $x \le 0$ follows from the same argument, we only deal with $x \ge 0$. Let us prove (2.6). For a given $m \in (0,1)$, we define $E_m(t)$ as the m-level set of $v(t,\cdot)$, namely

$$E_m(t) := \{x > 0 : v(t,x) = m\}.$$

We claim that there exist $M > 0$ and $T > 0$ such that

$$c_v t - \frac{3d}{c_v}\ln t - M \le \min E_m(t) \le \max E_m(t) \le c_v t - \frac{3d}{c_v}\ln t + M, \qquad \text{for all } t \ge T. \tag{4.22}$$

The upper bound in (4.22) is obtained by using the solution of $\partial_t\overline{v} = d\overline{v}_{xx} + r\overline{v}(1-\overline{v})$, starting from $\overline{v}(0,x) = v_0(x)$, as a super-solution. We refer to [22, Lemma 4.1], see also [2] and [12]. As for the lower bound in (4.22), it follows from [22, Lemma 4.5] which is based on an idea of [12]. A sketch of the proof is as follows. Let us a fix a small $\varepsilon > 0$ (this is necessary because of the bump phenomenon). By combining (4.5), Proposition 4.4 and (4.18)[3], we see that there exist $C > 0$, $\mu > 0$ and $T > 0$ such that

$$\sup_{|x|\ge\varepsilon t} u(t,x) \le Ce^{-\mu t}, \qquad \text{for all } t \ge T. \tag{4.23}$$

As a result, by setting $C_0 = rC$, we have

$$\partial_t v \ge dv_{xx} + v(r - rv - C_0 e^{-\mu t}), \quad \text{for all } t > 0, x > \varepsilon t.$$

[3]from which one can straightforwardly deduce that $\sup_{x\ge\varepsilon t} s(t,x) = s(t,\varepsilon t) = O\left(\frac{e^{-\frac{\varepsilon^2}{4}t}}{\sqrt{t}}\right)$.

The key idea, borrowed from [12], is then to linearize the above equation, and to consider

$$\partial_t w = dw_{xx} + w(r - C_0 e^{-\mu t}), \quad t > 0, x > \Gamma(t) := c_v t - \frac{3d}{c_v} \ln(t + t_0), \tag{4.24}$$

together with the Dirichlet boundary conditions $w(t, \Gamma(t)) = 0$ and a compactly supported initial datum $w(0, \cdot)$. Then, one can exactly reproduce the technical arguments of [22, Lemma 4.3 and 4.4], mainly borrowed from [12], to obtain the lower bound in (4.22). Last, by applying (4.22), we can reproduce the argument of [12, Section 4], see also [22, Proof of Theorem 2], to conclude that there exists a bounded function $\eta_* : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \sup_{x\geq 0} \left| v(t,x) - V_1 \left(x - c_v t + \frac{3d}{c_v} \ln t + \eta_*(t) \right) \right| = 0, \tag{4.25}$$

which, combined with (4.20), concludes the proof of (2.6). □

5 The bump phenomenon

In this section, we will provide a lower estimate for the solution of system (1.1) starting from a compactly supported initial datum, and prove Theorem 2.7 on the bump phenomenon.

5.1 Construction of the sub-solution

The goal of this subsection is to construct an adequate sub-solution in, roughly speaking, the region $|x| < c_v t$. More precisely, let $r_2 > r$ be given and define $c_v^{**} := 2\sqrt{dr_2} > c_v$. Let us fix $c_v < c_2 < c_v^{**}$. For $T > 0$, we will work in the domain (which is "expanding in time")

$$\Omega_2(T) := \{(t,x) \in (T, \infty) \times \mathbb{R} : |x| < c_2 t\}. \tag{5.1}$$

A key observation for the construction is the following: from Proposition 4.2, in the region $c_u t < |x| < c_v t$, we have $u + v \approx 1$, and therefore

$$\begin{cases} \partial_t u \approx u_{xx}, \\ \partial_t v \approx dv_{xx}. \end{cases}$$

We thus introduce $f = f(t,x)$ and $h = h(t,x)$ the solutions of the Cauchy problems

$$\begin{cases} \partial_t f = f_{xx}, \\ f(0,x) = f_0(x) := B_2 \mathbf{1}_{(-1,1)}(x), \end{cases} \qquad \begin{cases} \partial_t h = h_{xx}, \\ h(0,x) = h_0(x) := B_3 e^{-k|x|}, \end{cases} \tag{5.2}$$

and look for a sub-solution (U, V) in the form

$$\begin{cases} U(t,x) := g(t) f(t,x) - h(t,x), \\ V(t,x) := V_2(x - c_v^{**} t - \zeta_0) + V_2(-x - c_v^{**} t - \zeta_0) - 1 - U(t,x) + \dfrac{1}{t^{1+\theta}}, \end{cases} \tag{5.3}$$

where

$$g(t) := \exp \frac{1}{\delta (1+t)^\delta}.$$

Here, all parameters that will be determined below (namely B_2, B_3, k, ζ_0, θ, δ) are positive, $B_2 < 1$, $B_3 < 1$, while V_2 is the traveling wave solution with speed $c_v^{**} = 2\sqrt{dr_2}$ satisfying

$$\begin{cases} dV_2'' + c_v^{**} V_2' + r_2 V_2(1 - V_2) = 0, \\ V_2(-\infty) = 1,\ V_2(\infty) = 0. \end{cases} \tag{5.4}$$

It is well-known that $V_2' < 0$ and there exist $\lambda_2 > 0$ and $M_2 > 0$ such that

$$1 - V_2(\xi) \sim M_2 e^{\lambda_2 \xi} \ \text{ as } \xi \to -\infty. \tag{5.5}$$

Next, we shall provide some estimates which are based on the "Heat kernel expressions" of the solutions f and h of (5.2). Note that, in Lemma 5.1, $0 < B_2 < 1$ and $0 < B_3 < 1$ can be relaxed to $B_2 > 0$ and $B_3 > 0$.

Lemma 5.1. *Let $\delta > 0$ and $k > 0$ be given, and set $B_3 = \gamma B_2$ with some $\gamma > 0$. Then the following holds.*

(i) For any given $0 < j < k$,

$$h(t,x) \le \frac{B_3}{\sqrt{\pi}} \frac{k}{k^2 - j^2} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}, \qquad \text{for all } t > 0, |x| \le 2jt.$$

(ii) For any given $0 < j < k$ and $T > 0$, there exists $\gamma_1 > 0$ such that, if $0 < \gamma \le \gamma_1$, then, for all $B_2 > 0$,

$$g(t)f(t,x) - h(t,x) > 0, \qquad \text{for all } t \ge T, |x| \le 2jt.$$

(iii) There is $T^0 > 0$ such that, for all $B_2 > 0$,

$$g(t)f(t,x) - h(t,x) \le 0, \qquad \text{for all } t \ge T^0, |x| = 2kt.$$

Proof. Since $f(t,\cdot)$ and $h(t,\cdot)$ are even functions, it is enough to deal with $x \ge 0$. Recalling that $h_0(x) = B_3 e^{-k|x|}$, we have

$$h(t,x) = \frac{B_3}{\sqrt{4\pi t}} \left(\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} e^{ky} dy + \int_0^{+\infty} e^{-\frac{(x-y)^2}{4t}} e^{-ky} dy \right),$$

which can be recast, after some elementary computations,

$$h(t,x) = \frac{B_3}{\sqrt{\pi}} \left(e^{k^2 t - kx} \int_{\frac{2kt - x}{2\sqrt{t}}}^{+\infty} e^{-w^2} dw + e^{k^2 t + kx} \int_{\frac{2kt + x}{2\sqrt{t}}}^{+\infty} e^{-w^2} dw \right). \tag{5.6}$$

Now, recalling that $\int_X^{+\infty} e^{-w^2} dw \le \frac{e^{-X^2}}{2X}$ for any $X > 0$, the above expression implies that, for any $0 \le x \le 2jt < 2kt$,

$$h(t,x) \le \frac{B_3}{\sqrt{\pi}} e^{-\frac{x^2}{4t}} \left(\frac{\sqrt{t}}{2kt - x} + \frac{\sqrt{t}}{2kt + x} \right) \le \frac{B_3}{\sqrt{\pi}} \frac{k}{k^2 - j^2} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}},$$

which proves (i).

Recalling that $f_0(x) = B_2\mathbf{1}_{(-1,1)}(x)$, we have

$$f(t,x) = \frac{B_2}{\sqrt{4\pi t}}\int_{-1}^{1} e^{-\frac{(x-y)^2}{4t}}dy = \frac{B_2}{\sqrt{\pi}}\int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-w^2}dw. \tag{5.7}$$

Hence, from $g(t) \geq 1$, $B_3 = \gamma B_2$, (5.7) and (i), we deduce that, for all $0 \leq x \leq 2jt$ and $t \geq T$,

$$\begin{aligned} g(t)f(t,x) - h(t,x) &\geq \frac{B_2}{\sqrt{\pi}}\left(t^{-\frac{1}{2}}e^{-\frac{(x+1)^2}{4t}} - \gamma\frac{k}{k^2-j^2}t^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}\right) \\ &\geq \frac{B_2}{\sqrt{\pi}}t^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}\left(e^{-\frac{1}{4T}}e^{-j} - \gamma\frac{k}{k^2-j^2}\right), \end{aligned} \tag{5.8}$$

which is enough to prove (ii).

From (5.6), we have $h(t,2kt) \geq \frac{B_3}{\sqrt{\pi}}e^{-k^2t}\int_0^{+\infty} e^{-w^2}dw = \frac{B_3}{\sqrt{\pi}}e^{-k^2t}\frac{\sqrt{\pi}}{2}$. Hence, from $B_3 = \gamma B_2$ and (5.7), we deduce that, for $t \geq T^0 := \frac{1}{2k}$,

$$\begin{aligned} g(t)f(t,2kt) - h(t,2kt) &\leq \frac{B_2}{\sqrt{\pi}}\left(\|g\|_\infty t^{-\frac{1}{2}}e^{-\frac{(2kt-1)^2}{4t}} - \gamma\frac{\sqrt{\pi}}{2}e^{-k^2t}\right) \\ &\leq \frac{B_2}{\sqrt{\pi}}e^{-k^2t}\left(\|g\|_\infty e^k t^{-\frac{1}{2}} - \gamma\frac{\sqrt{\pi}}{2}\right), \end{aligned}$$

which is nonpositive, up to increasing T^0 if necessary. The proof of (iii) is complete. □

Remark 5.2. The statement (ii) in Lemma 5.1 guarantees that $U(t,x) = g(t)f(t,x) - h(t,x)$ is not a trivial sub-solution if γ is small enough.

Lemma 5.3. *There exists $C = C(k) > 0$ such that*

$$|f(t,x)| + |h(t,x)| \leq Ct^{-\frac{1}{2}}, \quad \textit{for all } t > 0, x \in \mathbb{R}, \tag{5.9}$$

and

$$|\partial_t f(t,x)| + |\partial_t h(t,x)| \leq Ct^{-\frac{3}{2}}, \quad \textit{for all } t > 0, x \in \mathbb{R}. \tag{5.10}$$

Proof. This proof is very classical. Denoting $G(t,x) := \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$, we have $f(t,x) = (G(t,\cdot) * f_0)(x)$, and thus

$$|f(t,x)| \leq \|G(t,\cdot)\|_\infty \|f_0\|_1 \leq C_G t^{-\frac{1}{2}}\|f_0\|_1,$$

with some $C_G > 0$. Also, we have $\partial_t f(t,x) = (\partial_t G(t,\cdot) * f_0)(x)$, and thus

$$|\partial_t f(t,x)| \leq \|\partial_t G(t,\cdot)\|_\infty \|f_0\|_1 \leq C'_G t^{-\frac{3}{2}}\|f_0\|_1,$$

with some $C'_G > 0$. Note that, $\|f_0\|_1 = 2B_2$, which implies C is independent on $B_2 < 1$ in (5.9) and (5.10).

Since $h_0 \in L^1(\mathbb{R})$, similar estimates hold for $h(t,x)$ and $\partial_t h(t,x)$, and $C = C(k)$ since $\|h_0\|_1 = \frac{2B_3}{k} \leq \frac{2}{k}$. □

We are now in the position to complete the construction of the sub-solution (U,V) in the form (5.3).

Proposition 5.4 (Sub-solutions)**.** *Let $0<\delta<\theta<\frac{1}{2}$ be given. Let us fix $k>0$, and set $B_3=\gamma B_2$ with $0<\gamma<1$.*

Then there exists $T^*>0$ such that, for all $0<B_2<1$ and $\zeta_0>0$, (U,V) is a sub-solution in the domain $\Omega_2(T^*)$ as defined in (5.1)*.*

Proof. Since $U(t,\cdot)$ and $V(t,\cdot)$ are even functions, it is enough to deal with $x\geq 0$. In other words, we work for $t\geq T$ (with $T>0$ to be selected) and $0\leq x<c_2t$, with $c_v<c_2<c_v^{**}$. For simplicity of notations, we shall use the shortcuts $\xi_\pm:=\pm x-c_v^{**}t-\zeta_0$. Since $\xi_-\leq -c_v^{**}t$ and $\xi_+\leq -(c_v^{**}-c_2)t$, it follows from $V_2'<0$ and (5.5) that there exist $C_->0$ and $C_+>0$ such that, for $T>0$ large enough,

$$1-V_2(\xi_-)\leq C_-e^{-\lambda_2 c_v^{**}t} \quad \text{and} \quad 1-V_2(\xi_+)\leq C_+e^{-\lambda_2(c_v^{**}-c_2)t}, \quad \text{for all } (t,x)\in\Omega_2^+(T), \tag{5.11}$$

where $\Omega_2^+(T):=\Omega_2(T)\cap(T,\infty)\times[0,\infty)$. Moreover, up to enlarging $T>0$ if necessary, there exists $0<\rho<\frac{1}{3}$ such that

$$0<1-V_2(\xi_\pm)\leq\rho, \quad \text{for all } (t,x)\in\Omega_2^+(T). \tag{5.12}$$

Some straightforward computations combined with (5.2) yield

$$\begin{aligned} N_1[U,V] &= g'f-(gf-h)(2-V_2(\xi_+)-V_2(\xi_-)-t^{-(1+\theta)}) \\ &\leq -(1+t)^{-(1+\delta)}gf+gft^{-(1+\theta)}+h\left(2-V_2(\xi_+)-V_2(\xi_-)-t^{-(1+\theta)}\right), \end{aligned}$$

since $2-V_2(\xi_+)-V_2(\xi_-)>0$. Thus, it follows from (5.11) that

$$N_1[U,V]\leq gf\left(-(1+t)^{-(1+\delta)}+t^{-(1+\theta)}\right)+h\left(C_-e^{-\lambda_2c_v^{**}t}+C_+e^{-\lambda_2(c_v^{**}-c_2)t}-t^{-(1+\theta)}\right).$$

Since $\delta<\theta$, it follows that, for $T>0$ large enough, $N_1[U,V]\leq 0$ in $\Omega_2^+(T)$.

Next, some straightforward computations combined with (5.2) and (5.4) yield

$$\begin{aligned} N_2[\bar{U},V] &= r_2V_2(\xi_+)(1-V_2(\xi_+))+r_2V_2(\xi_-)(1-V_2(\xi_-))+(1+t)^{-(1+\delta)}gf \\ &\quad +(d-1)(g\partial_tf-\partial_th)-(1+\theta)t^{-(2+\theta)} \\ &\quad -r(V_2(\xi_+)+V_2(\xi_-)-1-gf+h+t^{-(1+\theta)})(2-V_2(\xi_+)-V_2(\xi_-)-t^{-(1+\theta)}) \\ &= I_1+\cdots+I_5, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= rgf\left(2-V_2(\xi_+)-V_2(\xi_-)+\frac{1}{r}(1+t)^{-(1+\delta)}\right), \\ I_2 &:= (1-V_2(\xi_-))\left((r_2-r)V_2(\xi_-)+r\left(2-2V_2(\xi_+)-t^{-(1+\theta)}-h\right)\right), \\ I_3 &:= (1-V_2(\xi_+))\left((r_2-r)V_2(\xi_+)-rh\right), \\ I_4 &:= (d-1)(g\partial_tf-\partial_th), \\ I_5 &:= rt^{-(1+\theta)}\left(V_2(\xi_-)+2V_2(\xi_+)-2+t^{-(1+\theta)}-gf+h-\frac{1+\theta}{r}t^{-1}\right). \end{aligned}$$

Since $0<V_2<1$, we have $I_1\geq 0$. From $r_2>r$, (5.9) and (5.12), we have

$$I_2\geq(1-V_2(\xi_-))\left((r_2-r)(1-\rho)-rt^{-(1+\theta)}-rCt^{-\frac{1}{2}}\right)\geq 0,$$

up to enlarging $T > 0$ if necessary. Similarly, we obtain $I_3 \geq 0$. Last, from (5.12) and Lemma 5.3, we obtain

$$I_4 + I_5 \geq rt^{-(1+\theta)} \left(1 - 3\rho - \|g\|_\infty C t^{-\frac{1}{2}} - \frac{1+\theta}{r} t^{-1} \right) - C|d-1|(\|g\|_\infty + 1) t^{-\frac{3}{2}}.$$

Since $\theta < \frac{1}{2}$ and $0 < \rho < \frac{1}{3}$, we have $I_4 + I_5 \geq 0$ up to enlarging $T > 0$ if necessary. As a result, for $T > 0$ large enough, $N_2[U,V] \geq 0$ in $\Omega_2^+(T)$, and the proof of Proposition 5.4 is complete. □

Note that, time T^* in Proposition 5.4 is independent on $0 < B_2 < 1$ and $\zeta_0 > 0$, which leaves "some room" to reduce B_2 and to enlarge ζ_0 so that the "initial order" is suitable for the comparison principle to be applicable. We shall also need the suitable "order on the boundary of the domain", which will be obtained by choosing an appropriate k and Lemma 5.1 *(iii)*. More precisely, the following holds.

Proposition 5.5 (Second estimate on (u,v))**.** *Let $0 < \delta < \theta < \frac{1}{2}$ be given. Let us fix $k := \frac{c_2}{2} > 0$, and set $B_3 = \gamma B_2$ with $0 < \gamma < 1$.*

Then there exist $T^{**} > 0$, $0 < B_2 < 1$ and $\zeta_0 > 0$ such that

$$U(t,x) \leq u(t,x) \quad \text{and} \quad v(t,x) \leq V(t,x), \quad \text{for all } t \geq T^{**}, |x| \leq c_2 t,$$

where (U,V) is given by (5.3).

Proof. We aim at applying the comparison principle in $\Omega_2(T)$, as defined in (5.1), with a well-chosen $T > 0$. From Proposition 5.4, for any $T \geq T^*$, we are equipped with a sub-solution (U,V) for which $0 < B_2 < 1$ and $\zeta_0 > 0$ are arbitrary.

We now focus on $|x| = c_2 t$, $t \geq T$, that is, with a slight abuse of language, the boundary of $\Omega_2(T)$. We now set $T^{**} := \max(T^*, T^0)$, where $T^0 > 0$ is provided by Lemma 5.1 *(iii)*. In particular this implies (recall the choice $k = \frac{c_2}{2}$) that

$$U(t, \pm c_2 t) \leq 0 \leq u(t, \pm c_2 t), \quad \text{for all } t \geq T^{**}.$$

Next, recalling that $c_v < c_2 < c_v^{**}$, it follows from Proposition 4.2 that $v(t, \pm c_2 t) \to 0$ as $t \to \infty$. On the other hand, for any $t \geq T^{**}$,

$$V(t, \pm c_2 t) \geq V_2(-(c_v^{**} - c_2)t) + V_2(-(c_2 + c_v^{**})t) - 1.$$

As a result, up to enlarging T^{**} if necessary, one has

$$v(t, \pm c_2 t) \leq V(t \pm c_2 t), \quad \text{for all } t \geq T^{**}.$$

Last we focus on the initial data, namely $t = T^{**}$ and $|x| \leq c_2 T^{**}$. From (4.1), there exists $\varepsilon > 0$ such that

$$\min \left(\inf_{|x| \leq c_2 T^{**}} u(T^{**}, x), \inf_{|x| \leq c_2 T^{**}} 1 - v(T^{**}, x) \right) \geq \varepsilon > 0.$$

We now select $0 < B_2 < \frac{\varepsilon}{2\|g\|_\infty}$. From this choice, we have

$$U(T^{**}, x) \leq \|g\|_\infty B_2 \leq \frac{\varepsilon}{2} \leq u(T^{**}, x), \quad \text{for all } |x| \leq c_2 T^{**},$$

and, for all $|x| \leq c_2 T^{**}$,

$$V(T^{**},x) \geq 2V_2(-\zeta_0) - 1 - \|g\|_\infty B_2 \geq 2V_2(-\zeta_0) - 1 - \frac{\varepsilon}{2} \geq 1-\varepsilon,$$

by selecting $\zeta_0 > 0$ large enough, which implies

$$v(T^{**},x) \leq V(T^{**},x), \qquad \text{for all } |x| \leq c_2 T^{**}.$$

As a consequence, the comparison principle can be applied in $\Omega_2(T^{**})$, which concludes the proof of Proposition 5.5. □

5.2 Proof of Theorem 2.7

It remains to prove the bump phenomenon which, as explained in Section 2, is reserved to the critical competition case under consideration. Let $0 < \varepsilon < c_u$ be given and let us prove (2.8) and (2.9). Let us set $T^{**} > 0$ such that both Proposition 5.5 and Proposition 4.4 apply. In the sequel, we always consider $t \geq T^{**}$ and $0 \leq x \leq \varepsilon t$.

In particular, one has

$$g(t)f(t,x) - h(t,x) = U(t,x) \leq u(t,x) \leq \tilde{U}(t,x) = t^{-(k^*-\frac{1}{2})}(1-e^{-\tau t})s(t,x).$$

This estimate and (5.8) (with $j = \frac{\varepsilon}{2}$) yield the lower estimate in (2.8). On the other hand, Lemma 5.1 (i) (with $j = \frac{\varepsilon}{2}$) provides an upper bound of the form $t^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}$ in the case $d \leq 1$ (since then $\partial_t s = s_{xx}$) and of the form $t^{-\frac{1}{2}}e^{-\frac{x^2}{4dt}}$ in the case $d \geq 1$ (since then $\partial_t s = d s_{xx}$). This gives the upper estimate in (2.8).

Similarly, one obtains

$$U(t,x) - t^{-(1+\theta)} \leq 1 - V(t,x) \leq 1 - v(t,x) \leq 1 - \tilde{V}(t,x) \leq 2 - V_1(-c_v^* t) - V_1(-(c_v^* - \varepsilon)t) + \tilde{U}(t,x),$$

which gives the lower estimate in (2.9). On the other hand, we deduce from (4.8) that there exist $C_- > 0$ and $C_+ > 0$ such that, for all $0 \leq x \leq \varepsilon t$,

$$\begin{aligned} 1 - \tilde{V}(t,x) &\leq C_- e^{-\lambda_1 c_v^* t} + C_+ e^{-\lambda_1 (c_v^* - \varepsilon)t} + \tilde{U}(t,x) \\ &\leq C t^{-k^*} e^{-\frac{x^2}{4d^* t}} + \tilde{U}(t,x), \end{aligned}$$

with some $C > 0$ provided $\varepsilon > 0$ is chosen sufficiently small. This gives the upper estimate in (2.9) and concludes the proof of Theorem 2.7. □

A A result on the strong-weak competition system

In this Appendix, we consider the strong-weak competition system $(0 < a < 1 < b)$

$$\begin{cases} \partial_t u = u_{xx} + u(1-u-av), \\ \partial_t v = d v_{xx} + rv(1-v-bu), \end{cases} \tag{A.1}$$

supplemented with an initial datum (u_0, v_0) satisfying (2.5), for which we need a technical result, namely Lemma A.1, which is inspired by [22, Lemma 2.8] and the forthcoming work [25].

When $c_v = 2\sqrt{rd} > c_u = 2$, as proved in [9], the spreading properties are rather subtle: the rapid competitor v invades first at speed c_v and is then replaced by the strong competitor u at a speed $\mathscr{C}$ which can take two different values. To make this clear, we quote the following from [9] to which we refer for more details. First, the strong-weak system admits a *minimal* monotone traveling wave solution (c_{LLW}, U, V) with speed $2\sqrt{1-a} \leq c_{LLW} \leq 2$, defined as

$$\begin{cases} U'' + c_{LLW}U' + U(1 - U - aV) = 0, \\ dV'' + c_{LLW}V' + rV(1 - V - bU) = 0, \\ (U, V)(-\infty) = (1, 0),\ (U, V)(\infty) = (0, 1), \\ U' < 0,\ V' > 0. \end{cases}$$

Next define the decreasing function $f : [2\sqrt{1-a}, +\infty) \to (2\sqrt{a}, 2(\sqrt{1-a} + \sqrt{a})]$

$$f(c) := c - \sqrt{c^2 - 4(1-a)} + 2\sqrt{a} \quad \text{so that} \quad f^{-1}(\tilde{c}) := \frac{\tilde{c}}{2} - \sqrt{a} + \frac{2(1-a)}{\tilde{c} - 2\sqrt{a}}.$$

If $2\sqrt{rd} \in (2, f(c_{LLW}))$, then define

$$c_{nlp} := f^{-1}(2\sqrt{rd}) = \sqrt{rd} - \sqrt{a} + \frac{1-a}{\sqrt{rd} - \sqrt{a}} \in (c_{LLW}, 2).$$

Then we can precise

$$\mathscr{C} = \begin{cases} c_{nlp} & \text{if } 2 < c_v < f(c_{LLW}), \quad \text{(nonlocal pulling phenomenon)}, \\ c_{LLW} & \text{if } c_v \geq f(c_{LLW}). \end{cases}$$

We now state the result used in the proof of Proposition 4.4.

Lemma A.1. *Assume $dr > 1$ (i.e. $c_v > c_u$). Let $(u, v) = (u, v)(t, x)$ be the solution of the strong-weak competition system* (A.1) *starting from an initial datum $(u_0, v_0) = (u_0, v_0)(x)$ satisfying* (2.5)*. Then the following holds.*

(i) For any $c > \mathscr{C}$, there exist $C_1 > 0$, $\nu_1 > 0$, $T_1 > 0$ such that

$$\sup_{|x| \geq ct} u(t, x) \leq C_1 e^{-\nu_1 t}, \quad \text{for all } t \geq T_1.$$

(ii) For any c_1 and c_2 with $\mathscr{C} < c_1 < c_2 < c_v$, there exist $C_2 > 0$, $\nu_2 > 0$, $T_2 > 0$ such that

$$\sup_{c_1 t \leq |x| \leq c_2 t} v(t, x) \geq 1 - C_2 e^{-\nu_2 t}, \quad \text{for all } t \geq T_2.$$

Proof. Let us briefly start with (i). If $c > c_u$ then the conclusion is clear by the same argument as in Proposition 4.2. Since $c_v > c_u$, it thus suffices to consider the case $\mathscr{C} < c < c_v$. In this case, the conclusion is already included in [9, Proposition 1.5] and the proof of [9, Section 3.2.3, Theorem 1.1]. We do not present the full details but only emphasize that a key tool is, for any small $\delta > 0$, the *minimal* monotone traveling wave of the perturbed system

$$\begin{cases} U'' + cU' + U(1 + \delta - U - aV) = 0, \\ dV'' + cV' + rV(1 - 2\delta - V - bU) = 0, \\ (U, V)(-\infty) = (1 + \delta, 0),\ (U, V)(\infty) = (0, 1 - 2\delta), \\ U' < 0,\ V' > 0. \end{cases}$$

Let us now turn to (ii) (which is the estimate we need in the proof of Proposition 4.4) for which the above perturbation argument seems unapplicable. Let $\mathscr{C} < c_1 < c_2 < c_v$ be given. We only deal with $x \geq 0$. From [9, Theorem 1.1] we know

$$\lim_{t\to\infty} \sup_{c_1 t \leq x \leq c_2 t} \Big(u(t,x) + |1 - v(t,x)| \Big) = 0.$$

From this and (i), one can choose $\varepsilon > 0$ small enough and $T_0 \gg 1$ such that

$$0 < u(t,x) \leq C_1 e^{-\nu_1 t}, \quad v(t,x) > 1 - \varepsilon, \quad \text{for all } t \geq T_0, x \in [c_1 t, c_2 t].$$

From the v-equation in (A.1), we have

$$v_t \geq d v_{xx} + r(1-\varepsilon)(1-v) - rbC_1 v e^{-\nu_1 t}, \quad \text{for all } t \geq T_0, x \in [c_1 t, c_2 t]. \tag{A.2}$$

Defining

$$\tilde{v}(t,x) := v(t, x + \tilde{c}t), \quad \tilde{c} := \frac{c_1 + c_2}{2}, \tag{A.3}$$

it follows from (A.2) that

$$\tilde{v}_t \geq d\tilde{v}_{xx} + \tilde{c}\tilde{v}_x + r(1-\varepsilon)(1-\tilde{v}) - rbC_1 \tilde{v} e^{-\nu_1 t}, \quad \text{for all } t \geq T_0, x \in [-c_3 t, c_3 t],$$

where $c_3 := \frac{c_2 - c_1}{2}$.

To estimate $\tilde{v}$, for any $T \geq T_0$, we define

$$\alpha(t) := 1 + \frac{bC_1}{1-\varepsilon} e^{-\nu_1 (t+T)}, \quad \text{for all } t \geq 0.$$

Up to enlarging $T > 0$ if necessary, we may assume $\alpha(0) < \frac{1}{1-\varepsilon}$. Now, let us first consider the auxiliary problem

$$\begin{cases} \varphi_t = d\varphi_{xx} + \tilde{c}\varphi_x + r(1-\varepsilon)[1 - \alpha(t)\varphi], & t > 0, \ -c_3 T < x < c_3 T, \\ \varphi(t, \pm c_3 T) = 1 - \varepsilon, & t > 0, \\ \varphi(0,x) = 1 - \varepsilon, & -c_3 T \leq x \leq c_3 T. \end{cases} \tag{A.4}$$

Letting

$$\Phi(t,x) := e^{Q(t)}[\varphi(t,x) - 1 + \varepsilon], \quad Q(t) := r(1-\varepsilon)t - \frac{rbC_1}{\nu_1} e^{-\nu_1 (t+T)},$$

so that $Q'(t) = r(1-\varepsilon)\alpha(t)$, it follows from (A.4) that

$$\begin{cases} \Phi_t = d\Phi_{xx} + \tilde{c}\Phi_x + r(1-\varepsilon)e^{Q(t)}(1 - (1-\varepsilon)\alpha(t)), & t > 0, \ -c_3 T < x < c_3 T, \\ \Phi(t, \pm c_3 T) = 0, & t > 0, \\ \Phi(0,x) = 0, & -c_3 T \leq x \leq c_3 T. \end{cases} \tag{A.5}$$

Up to a rescaling, we may assume $d = 1$ so that (A.5) is very comparable to [15, problem (3.12)] on which we now rely. Denoting $G_1(t,x,z)$ the Green function of [15, page 53] (with obvious changes of constants), we obtain the analogous of [15, (3.14)], namely

$$\Phi(t,x) \geq r(1-\varepsilon) \int_0^t e^{Q(s)} (1 - (1-\varepsilon)\alpha(s)) \left(\int_{-c_3 T}^{c_3 T} G_1(t-s, x, z) dz \right) ds,$$

for all $t > 0$, $-c_3 T < x < c_3 T$. Next, for any small $0 < \delta < 1$, we define

$$D_\delta := \left\{(t, x) \in \mathbb{R}^2 : 0 < t < \delta^2 c_3 T, |x| < (1-\delta) c_3 T\right\}.$$

From the same process used in [15, pages 54-55], there exist $C_3, C_4 > 0$ such that the following lower estimate holds

$$\Phi(t, x) \geq r(1-\varepsilon)(1-(1-\varepsilon)\alpha(0))(1 - C_3 e^{-C_4 T}) \int_0^t e^{Q(s)} ds, \quad \text{for all } (t, x) \in D_\delta,$$

resulting in

$$\varphi(t, x) \geq \Psi(t)(1 - C_3 e^{-C_4 T})(1-(1-\varepsilon)\alpha(0)) + 1 - \varepsilon, \quad \text{for all } (t, x) \in D_\delta, \tag{A.6}$$

where $\Psi(t) := r(1-\varepsilon) e^{-Q(t)} \int_0^t e^{Q(s)} ds$. Denoting $K = \frac{rbC_1}{\nu_1}$, we have

$$\begin{aligned} \Psi(t) &\geq r(1-\varepsilon) e^{-Q(t)} \int_0^t e^{r(1-\varepsilon)s} e^{-Ke^{-\nu_1 T}} ds \\ &= e^{-r(1-\varepsilon)t} e^{Ke^{-\nu_1(t+T)}} e^{-Ke^{-\nu_1 T}} \int_0^t r(1-\varepsilon) e^{r(1-\varepsilon)s} ds \\ &= e^{Ke^{-\nu_1 T}(e^{-\nu_1 t} - 1)}(1 - e^{-r(1-\varepsilon)t}). \end{aligned}$$

Inserting this into (A.6) and using $e^y \geq 1 + y$ for all $y \in \mathbb{R}$, we have, for all $(t, x) \in D_\delta$,

$$\begin{aligned} \varphi(t, x) &\geq e^{Ke^{-\nu_1 T}(e^{-\nu_1 t} - 1)}(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1-(1-\varepsilon)\alpha(0)) + 1 - \varepsilon \\ &\geq (1 - Ke^{-\nu_1 T}(1 - e^{-\nu_1 t}))(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1-(1-\varepsilon)\alpha(0)) + 1 - \varepsilon \\ &\geq (1 - Ke^{-\nu_1 T})(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1-(1-\varepsilon)\alpha(0)) + 1 - \varepsilon. \end{aligned}$$

Letting

$$I_1 := 1 - Ke^{-\nu_1 T}, \quad I_2(t) := 1 - e^{-r(1-\varepsilon)t}, \quad I_3 := 1 - C_3 e^{-C_4 T},$$

we get

$$\varphi(t, x) \geq I_1 I_2(t) I_3 + (1-\varepsilon)(1 - I_1 I_2(t) I_3 \alpha(0)), \quad \text{for all } (t, x) \in D_\delta.$$

Now observe that $I_1 I_2(t) I_3 \leq I_1 I_2(\delta^2 c_3 T) I_3$. Furthermore some straightforward computations show that, if

$$r(1-\varepsilon)\delta^2 c_3 < \nu_1, \tag{A.7}$$

then $I_1 I_2(\delta^2 c_3 T) I_3 \alpha(0) \leq 1$ up to enlarging $T > 0$ if necessary. As a result, for all $(t, x) \in D_\delta$,

$$\varphi(t, x) \geq I_1 I_2(t) I_3 \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)t},$$

with some $K_1, K_2 > 0$. The last inequality holds since we can always choose $\nu_1 < C_4$. As a conclusion, we have

$$\varphi(t, x) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)t}, \quad \text{for all } (t, x) \in D_\delta, \tag{A.8}$$

provided that $\delta > 0$ is sufficiently small for (A.7) to hold and $T > 0$ is sufficiently large.

In particular (A.8) implies that, for all $|x| \leq (1-\delta) c_3 T$,

$$\varphi\left(\delta^2 c_3 T, x\right) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)\delta^2 c_3 T} \geq 1 - (K_1 + K_2) e^{-r(1-\varepsilon)\delta^2 c_3 T}, \tag{A.9}$$

in virtue of (A.7). On the other hand, we know from the comparison principle that $\tilde{v}(t+T,x) \geq \varphi(t,x)$ for $t \geq 0$ and $|x| \leq c_3 T$, which together with (A.9) implies that

$$\tilde{v}\left(\delta^2 c_3 T + T, x\right) \geq 1 - (K_1+K_2)e^{-r(1-\varepsilon)\delta^2 c_3 T}, \quad \text{for all } |x| \leq (1-\delta)c_3 T.$$

We further take $t = (\delta^2 c_3 + 1)T$, which yields

$$\tilde{v}(t,x) \geq 1 - C_2 e^{-\nu_2 t}, \quad \text{for all large } t \text{ and } |x| \leq \tfrac{(1-\delta)c_3}{1+c_3\delta^2} t,$$

where $C_2 := K_1 + K_2$ and $\nu_2 := \frac{r(1-\varepsilon)\delta^2 c_3}{1+c_3\delta^2} > 0$. Recalling that $\tilde{v}(t,x) = v(t, x+\tilde{c}t)$ with $\tilde{c} = \frac{c_1+c_2}{2}$, that $c_3 = \frac{c_2-c_1}{2}$ and since $\delta > 0$ can be chosen arbitrarily small, the above estimate completes the proof of (ii). □

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References

[1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. Math., 30 (1978), 33–76.

[2] M. D. Bramson, Convergence of solutions of the Kolmogorov equation to travelling waves, Mem. Amer. Math. Soc., 44 (1983), no. 285, iv+190 pp.

[3] C. Carrère, Spreading speeds for a two-species competition-diffusion system. J. Differential Equations, 264 (2018), 2133–2156.

[4] A. Ducrot, T. Giletti and H. Matano, Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations, Trans. Amer. Math. Soc. 366 (2014), no. 10, 5541–5566.

[5] A. Ducrot, T. Giletti and H. Matano, Spreading speeds for multidimensional reaction-diffusion systems of the prey-predator type, Calc. Var. Partial Differential Equations, 58 (2019), Paper No. 137, 34 pp.

[6] R. A. Fisher, The wave of advance of advantageous genes, Ann. Eugen., 7 (1937), 335–369.

[7] R.A. Gardner, Existence and stability of traveling wave solutions of competition models: a degree theoretic approach, J. Differential Equations, 44 (1982), 343–364.

[8] L. Girardin, The effect of random dispersal on competitive exclusion-a review, Mathematical Biosciences, 318 (2019), 108271.

[9] L. Girardin, K.-Y. Lam, Invasion of an empty habitat by two competitors: spreading properties of monostable two-species competition-diffusion systems, Proc. Lond. Math. Soc., 119 (2019), 1279–1335.

[10] L. Girardin and G. Nadin, Travelling waves for diffusive and strongly competitive systems: relative motility and invasion speed, European J. Appl. Math., 26 (2015), 521–534.

[11] J.-S. Guo and Y.-C. Lin, The sign of the wave speed for the Lotka-Volterra competition-diffusion system, Comm. Pure Appl. Anal., 12 (2013), 2083–2090.

[12] F. Hamel, J. Nolen, J. M. Roquejoffre and L. Ryzhik, A short proof of the logarithmic Bramson correction in Fisher-KPP equations, Netw. Heterog. Media., 8 (2013), 261–289.

[13] Y. Kan-On, Parameter dependence of propagation speed of travelling waves for competition-diffusion equations, SIAM J. Math. Anal., 26 (1995), 340–363.

[14] Y. Kan-On, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, Nonlinear Anal. TMA, 28 (1997), 145-164.

[15] Y. Kaneko and H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection-diffusion equation, J. Math. Anal. Appl, 428 (2015), 43–76.

[16] A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem, Bull. Moscow State Univ. Ser. A: Math. and Mech., 1 (1937), 1–25.

[17] M.A. Lewis, B. Li and H.F. Weinberger, Spreading speeds and the linear conjecture for two-species competition models, J. Math. Biol., 45 (2002) 219–233.

[18] M.A. Lewis, B. Li and H.F. Weinberger, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci., 196 (2005), 82–98.

[19] G. Lin, W.-T. Li, Asymptotic spreading of competition diffusion systems: the role of interspecific competitions. European J. Appl. Math., 23 (2012), 669–689.

[20] Q. Liu, S. Liu, K.-Y. Lam, Asymptotic spreading of interacting species with multiple fronts I: A geometric optics approach, Discrete Cont. Dyn. Syst. Ser. A, 40 (2020), 3683–3714.

[21] Q. Liu, S. Liu, K.-Y. Lam, Asymptotic spreading of interacting species with multiple fronts II: Exponentially decaying initial data, preprint.

[22] R. Peng, C. Wu and M. Zhou, Sharp estimates for the spreading speeds of the Lotka-Volterra diffusion system with strong competition, Ann. Inst. H. Poincaré Anal. Non Linéaire, 38 (2021), 507–547.

[23] M. Rodrigo, M. Mimura, Exact solutions of a competition-diffusion system, Hiroshima Math. J., 30 (2000), 257–270.

[24] M. M. Tang, P. Fife, Propagating fronts for competing species equations with diffusion, Arch. Rational Mech. Anal. 73 (1980), no. 1, 69–77.

[25] C.-H. Wu, D. Xiao and M. Zhou, Lotka-Volterra diffusion systems: the strong-weak type, in preparation.