Infinitely many saturated travelling waves for epidemic models with distributed-contacts

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Abstract

We consider an epidemic model with distributed-contacts. When the contact kernel concentrates, one formally reaches a very degenerate Fisher-KPP equation with a diffusion term that is not in divergence form. We make an exhaustive study of its travelling waves. For every admissible speed, there exists not only a non-saturated (smooth) wave but also infinitely many saturated (sharp) ones. Furthermore their tails may differ from what is usually expected. These results are thus in sharp contrast with their counterparts on related models.

KEYWORDS: epidemic model, degenerate Fisher-KPP equation not in divergence form, infinitely many travelling waves, unusual tails.

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1 Introduction

Starting from a SI epidemic model with so-called distributed-contacts, we show the relevance of a very degenerate (since not in divergence form) reaction-diffusion equation of the Fisher-KPP type for the infectious density $I = I(t, x)$, namely

$$\partial_t I = (1 - I)\Delta I + I(1 - I), \quad t > 0,\, x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 1$. We focus on the existence and properties of planar travelling waves for (1). We show that, for any admissible speed, there exists not only a non-saturated (smooth) wave but also infinitely many saturated ones. This is in sharp contrast with existing results on related models. The tails of these waves also exhibit unexpected behavior.

Compartmental epidemics models were introduced in the seminal work of Kermack and McKendrick [28]. The simplest SI model consists of a ODE system for $S = S(t)$, $I = I(t)$,

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the number of Susceptible and Infectious, uses a law of mass action and writes

$$\begin{cases} \frac{dS}{dt} = -\beta SI & t>0, \\ \frac{dI}{dt} = \beta SI - \mu I & t>0, \end{cases} \tag{2}$$

where $\beta > 0$ is the constant transmission rate, $\mu \geq 0$ the constant recovery rate. However spatial effects, which have a very determining effect on the propagation of epidemics, are neglected in this simple model. To fill this gap, Kendall [27] and Mollison [32] have allowed (see also [35], [16], [31]) spatially distributed-contacts between individuals. In this framework, in absence of recovery ($\mu = 0$) and letting $\beta = 1$ up to changing the time scale, $S = S(t,x)$, $I = I(t,x)$ solve the integro-differential system

$$\begin{cases} \partial_t S = -S \displaystyle\int_{\mathbb{R}^N} K(x,y) I(t,y)\, dy & t>0,\ x \in \mathbb{R}^N, \\ \partial_t I = S \displaystyle\int_{\mathbb{R}^N} K(x,y) I(t,y)\, dy & t>0,\ x \in \mathbb{R}^N, \end{cases} \tag{3}$$

with $K(x,y)$ denoting the density function for the proportion of infectious at position y that contact susceptibles at position x. Since $S+I$ is independent of time, say equal to 1, we may rewrite

$$\partial_t I = (1-I)(J*I - I) + I(1-I), \quad t>0,\ x \in \mathbb{R}^N, \tag{4}$$

where we have assumed $K(x,y) = J(x-y)$ with J a probability density on $\mathbb{R}^N$, meaning that contacts are *homogeneous* in space. However, in some situations, nonlocal diffusion operators can be approximated by local ones. This fact is well-known, see e.g. [13, Chapter VI, subsection 6.4], and can be understood from a simple formal Taylor expansion. Indeed, assuming that J is radial (note that weaker conditions such as component-wise symmetry would be enough) and has a finite second moment $J_2 := \int_{\mathbb{R}^N} |z|^2 J(z)\, dz < +\infty$, replacing the kernel $J(z)$ by the focused kernel

$$J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right), \quad 0 < \varepsilon \ll 1,$$

we see that

$$\begin{aligned} (J_\varepsilon * I - I)(t,x) &= \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} J\left(\frac{x-y}{\varepsilon}\right) (I(t,y) - I(t,x))\, dy \\ &= \int_{\mathbb{R}^N} J(z)(I(t,x-\varepsilon z) - I(t,x))\, dz \\ &\approx \varepsilon^2 \frac{J_2}{2N} \Delta I(t,x), \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

For a rigorous statement one may refer to [3, Theorem 1.24]. Let us also note that the case of nonlocal *heterogeneous* diffusion has recently been explored in [1]. Hence, starting from a SI model with distributed-contacts, and formally passing to the focusing kernel limit, we have reached a degenerate Fisher-KPP equation of the form (1).

Since the introduction of spatial effects in epidemics models, the issue of travelling wave solutions has attracted much attention. We may refer, among many others, to the works [6], [12], [7], [4], [35], [16], or [26]. Very recently, a renewed interest for these models, both from the modelling and the mathematical analysis point of view, has emerged. Let us mention for instance the focus on heterogeneities [21], [18], [19], the effect of fast lines of diffusion [8], the spread of epidemics on graphs [11], the spread of several variants of a disease [20], [14], [24], etc.

Planar travelling wave solutions are particular solutions (of reaction-diffusion equations) describing the transition at a constant speed c from one stationary solution to another one. In the case of the classical Fisher-KPP equation $\partial_t I = \Delta I + I(1-I)$, it has long been known [25], [29], [5], that they exist if and only if $c \geq c^* := 2$. Moreover, they are very accurate to describe the long time behavior of the Cauchy problem, the tails of the initial data selecting the speed $c \geq c^*$. In particular, the minimal speed $c^* = 2$ corresponds to the so-called *spreading speed* of propagation for the Cauchy problem with compactly supported data. In the Fisher-KPP framework, let us also mention the construction of travelling wave solutions in presence of density-dependent diffusion [22], possibly degenerate [33, 34], [30], [17]. However, much less is known when the equation is, as (1), not in divergence form. We mention the works [37], [38, 39], mainly concerned with diffusion of the form $I^m \Delta I$ (or variants involving the p-Laplacian), meaning that singularities are caused by $I \equiv 0$ (the unstable equilibrium of the logistic equation), which is in sharp contrast with (1) where singularities are caused by $I \equiv 1$ (the stable equilibrium of the logistic equation).

In this work, we are thus concerned with the existence of travelling wave solutions for (1). For any admissible speed $c \geq c^* := 2$, we will show that there is a travelling wave solution with value in $(0,1)$, which is quite expected. However, the equation being not in divergence form, we will also show that, for any admissible speed $c \geq c^* := 2$, there are infinitely many travelling waves saturating at value 1 on some semi-infinite interval. This phenomenon is quite unusual and raises many questions regarding the Cauchy problem, in particular concerning its well-posedness and long time behavior.

2 Main results

Let us propose a notion of weak solution for equation (1). First, we impose the bound $I \leq 1$ for the equation not to become anti-diffusive. Next, we rewrite the diffusion term $(1-I)\Delta I$ as $\nabla \cdot ((1-I)\nabla I) + |\nabla I|^2$ and, classically, multiply by test functions and integrate by parts, see [9] for a similar formulation.

Definition 2.1 (Weak solution). *A weak solution of* (1) *is a function* $I \in L^\infty(\mathbb{R} \times \mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}, H^1_{loc}(\mathbb{R}^N))$ *such that* $I \leq 1$ *and*

$$\int_{\mathbb{R}\times\mathbb{R}^N} \left(I\partial_t \varphi - (1-I)\nabla I \cdot \nabla \varphi + |\nabla I|^2 \varphi + I(1-I)\varphi \right)(t,x)\, dtdx = 0, \qquad (5)$$

for all compactly supported $\varphi \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N)$.

As far as the Cauchy problem is concerned, the well-posedness (existence and uniqueness) is a delicate issue because of the non divergence form of the equation, see [2], [36], [15], [10], [9]. In [36] Ughi, in the one dimensional case, has worked in the framework of solutions that are *Lipschitz* in space. Using a formulation in the spirit of Definition 2.1, she proves existence by some approximation/regularization procedure, but highlights a non uniqueness result. Next, she proposes another weak formulation based on

$$\partial_t(\log(1-I)) = \Delta(1-I) - I,$$

obtained by first dividing the equation by $1-I$. It turns out that this improved formulation ensures uniqueness. However, the support of $1-I$ does not depend on time. It is challenging to define a notion of weak solution for the Cauchy problem not only ensuring uniqueness but also allowing this support to move, hence capturing the saturated fronts constructed in the present paper. We underline that these saturated fronts are not Lipschitz in space and this fact should play a key role. We hope to address this issue in a future work.

Now we define the notion of a planar travelling wave for (1).

Definition 2.2 (Travelling wave profile). *A travelling wave profile is a couple (z^*, u) made of a $-\infty \leq z^* < +\infty$ and a function $u : (z^*, +\infty) \to (0,1)$ of class C^2 on $(z^*, +\infty)$, such that $u(z^*) = 1$, $u(+\infty) = 0$.*

(i) If $z^* = -\infty$, the wave is said to be **non-saturated**.

(ii) If $z^* \neq -\infty$, the wave is said to be **saturated**.

Definition 2.3 (Travelling wave solution to (1)). *A travelling wave solution to* (1) *propagating in the direction $e \in \mathbb{S}^{N-1}$ is a triplet (c, z^*, u), made of a speed $c \in \mathbb{R}$ and a travelling wave profile (z^*, u), such that*

$$I(t,x) := \begin{cases} u(x \cdot e - ct) & \text{if } x \cdot e - ct > z^*, \\ 1 & \text{if } x \cdot e - ct \leq z^* \end{cases} \tag{6}$$

is a weak solution to (1) *in the sense of Definition 2.1.*

Our main result is the following exhaustive description of travelling waves.

Theorem 2.4 (Travelling waves). *There is no travelling wave solution to* (1) *with $c < 2$. Next, let $c \geq 2$ be fixed. Then there exists a non-saturated and infinitely many saturated waves normalized by $u(0) = \frac{1}{2}$. More precisely, defining*

$$\lambda^- := \frac{c - \sqrt{c^2 - 4}}{2}, \quad \lambda^+ := \frac{c + \sqrt{c^2 - 4}}{2},$$

there are constants $C_i > 0$ such that for any $0 < \varepsilon \ll 1$ the following hold.

(i) There exists a non-saturated wave whose profile is denoted u_{NS}. It satisfies

$$C_1 \, e^{-\lambda^- z} \leq u_{NS}(z) \leq C_2 \, e^{-(\lambda^- - \varepsilon) z}, \qquad \text{as } z \to +\infty, \tag{7}$$

$$C_3 \, e^{\frac{1}{c} z} \leq 1 - u_{NS}(z) \leq C_4 \, e^{\frac{1}{c+\varepsilon} z}, \qquad \text{as } z \to -\infty. \tag{8}$$

Last, there is $z_0 > 0$ such that $u'' < 0$ on $(-\infty, z_0)$ while $u'' > 0$ on $(z_0, +\infty)$.

(ii) There exists infinitely many saturated waves whose profiles are denoted u_S. They satisfy

$$1 - u_S(z) \sim -c(z - z^*) \log(z - z^*), \qquad \text{as } z \searrow z^*. \tag{9}$$

Among them

(a) there are infinitely many ones for which there are $z_1 < 0 < z_2$ such that $u_S'' > 0$ on (z^*, z_1), $u_S'' < 0$ on (z_1, z_2), $u_S'' > 0$ on $(z_2, +\infty)$. They satisfy

$$C_5 \, e^{-\lambda^- z} \leq u_S(z) \leq C_6 \, e^{-(\lambda^- - \varepsilon) z}, \qquad \text{as } z \to +\infty. \tag{10}$$

(b) *there is one such that* $u_S'' > 0$ *on* $(z^*, 0) \cup (0, +\infty)$ *and* $u_S''(0) = 0$*. It satisfies* (10).

(c) *there are infinitely many ones such that* $u_S'' > 0$ *on* $(z^*, +\infty)$*. Infinitely many of them satisfy*

$$C_7\, e^{-(\lambda^- + \varepsilon) z} \leq u_S(z) \leq C_8\, e^{-(\lambda^- - \varepsilon) z}, \qquad \text{as } z \to +\infty. \tag{11}$$

At least one other satisfies

$$C_9\, e^{-(\lambda^+ + \varepsilon) z} \leq u_S(z) \leq C_{10}\, e^{-\lambda^+ z}, \qquad \text{as } z \to +\infty. \tag{12}$$

The above results are in very sharp contrast with previous results on related problems.

Indeed, and first of all, for a given admissible speed $c \geq 2$, *two* different kind of waves, non-saturated and saturated, co-exist. This is already very unusual and, of the top of it, the saturated waves are *infinitely many*.

Also, from our construction, any two of the waves of Theorem 2.4 cross each other exactly once, namely at $z = 0$. Furthermore, see the shooting argument with α in (33),

$$u_{NS} > u_S \text{ of type } (a) > u_S \text{ of type } (b) > u_S \text{ of type } (c) \quad \text{ on } (0, +\infty),$$

and reversed order on $(-\infty, 0)$. Also from the shooting argument we use in Section 3, the points z^* where the saturated waves reach 1 describe $(-\infty, z^*_{max}]$ for some $z^*_{max} < 0$, see Figure 1 for a picture of the situation.

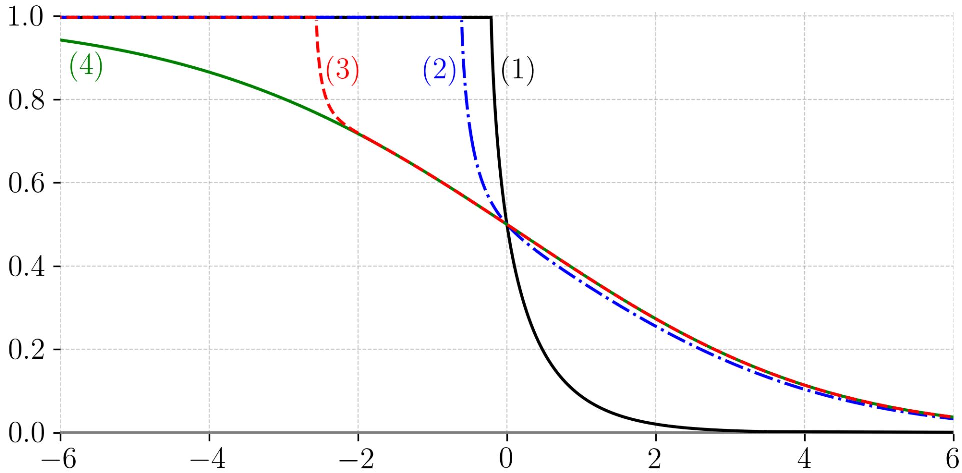

Figure 1: Some travelling waves of Theorem 2.4 (for a given speed $c = 2.1$). The profile (1), (2) and (3) are saturated waves. Specifically, (1) and (2) are profiles of type (c) (convex), whereas (3) is a profile of type (a) (convex-concave-convex). Profile (4) corresponds to the unique non-saturated wave.

Another striking phenomenon concerns the *right tail* of these waves. Indeed, when $c > 2$ (supercritical speeds), (7) and (12) indicate that the decay towards zero of *some* u_{NS} and u_S are not the same: since then $\lambda^- < \lambda^+$, we even have

$$\text{if } \ c > 2 \ \text{ then } \ u_S(z) \ll u_{NS}(z) \text{ as } z \to +\infty. \tag{13}$$

Let us recall that if we linearize at $z = +\infty$ the travelling wave equation for the classical Fisher-KPP equation, namely $u'' + cu' + u(1-u) = 0$, we get $u'' + cu' + u = 0$, which obviously admits two exponential solutions $e^{-\lambda^- z}$ and $e^{-\lambda^+ z}$, connecting $+\infty$ as $z \to -\infty$ to 0 as $z \to +\infty$. However, when considering the full nonlinear equation with $u(-\infty) = 1$, the right tail is of the magnitude $e^{-\lambda^- z}$, namely

$$u_{classical}(z) \sim \begin{cases} Ce^{-\lambda^- z} & \text{if } c > 2, \\ Cze^{-z} & \text{if } c = 2, \end{cases} \quad \text{as } z \to +\infty.$$

In other words, the behavior $e^{-\lambda^+ z}$ is erased by the *a priori* condition $u' = 0$ at $u = 1$ in the classical Fisher-KPP equation but can still persist in the degenerate case under consideration, see (12). In the latter case the asymptotic behavior at $u = 1$ differs, see Lemma 3.1.

Note that (13) is true for the u_S satisfying (12) (at least one), but this may still hold for any of the constructed u_S's. Understanding this would require very refined asymptotics of $h(r)$ as $r \to 0$ (see Section 3 for the definition of h and details) and is left as an open question.

Similarly, determining if there is *exactly one* or *infinitely many* waves satisfying (12) is far from obvious and would require refined estimates of $h(r)$ as $r \to 0$ (see Section 3 for details).

The *left tail* of the non-saturated wave u_{NS} is also unusual. Indeed, in the classical Fisher-KPP equation, as seen from linearizing at $z = -\infty$ and as well-known,

$$1 - u_{classical}(z) \sim Ce^{\frac{-c+\sqrt{c^2+4}}{2} z}, \quad \text{as } z \to -\infty.$$

However, since $\frac{1}{c} > \frac{-c+\sqrt{c^2+4}}{2}$, (8) indicates that the left tail of the non-saturated wave is much lighter, namely

$$1 - u_{NS}(z) \ll 1 - u_{classical}(z), \quad \text{as } z \to -\infty.$$

The reason for (8) can be understood as follows: as $z \to -\infty$, meaning $u \to 1$, the degenerate travelling wave equation $(1-u)u'' + cu' + u(1-u) = 0$ is, at least formally, approximated by the first order ODE $cu' + (1-u) = 0$ so that $1 - u(z)$ decays like $e^{\frac{1}{c} z}$.

Obviously, these various phenomena are induced by the non-divergence form of the equation and raise further questions not only on the well-posedness (as mentioned above) but also on the long time behavior of the Cauchy problem associated with (1). Indeed, the exponential right tail of a front-like initial datum cannot decide alone the spreading speed, as in the classical case. We believe that if $0 < u_0 < 1$ on $\mathbb{R}$ then the spreading speed is that of u_{NS} having the right tail "matching" with that of u_0. However, if $u_0 \equiv 1$ on some interval, the picture is less clear. This would deserve further investigations.

The rest of the paper is devoted to the proof of Theorem 2.4. The strategy consists in transforming in subsection 3.1 the problem into a first order singular ODE Cauchy problem, studied in details in subsections 3.2 and 3.3. Last, we conclude in subsection 3.4.

3 Travelling wave solutions

By formally plugging the ansatz $I(t, x) = u(x \cdot e - ct)$ into (1), it is clear that the profile (z^*, c, u) should satisfy the equation

$$(1-u)u'' + cu' + u(1-u) = 0, \quad \text{on } (z^*, +\infty). \tag{14}$$

We start with the following basic facts about (14).

Lemma 3.1 (A priori facts)**.** *Let (z^*, u) be a travelling wave profile in the sense of Definition 2.2 which, for some $c \in \mathbb{R}$, satisfies* (14). *Then*

(i) $u' < 0$ *on* $(z^*, +\infty)$.

(ii) $u'(+\infty) = 0$, $(1 - u(z))u'(z) \to 0$ *as* $z \to z^*$, *and* $u' \in L^2(z^*, +\infty)$.

(iii) $c > 0$.

Proof. Since u takes value in $(0, 1)$, it follows from equation (14) that any critical point z is such that $u''(z) < 0$. Hence if there is $z_0 \in (z^*, +\infty)$ such that $u'(z_0) = 0$ the boundary condition $u(z^*) = 1$ cannot be satisfied. This proves (i).

Since $(1 - u)u'' = ((1 - u)u')' + u'^2$, integrating the equation over (z, t) yields

$$(1 - u(z))u'(z) = (1 - u(t))u'(t) + c(u(t) - u(z)) + \int_z^t (u'^2 + u(1 - u))(s)\, ds. \tag{15}$$

Since $u'^2 + u(1 - u) > 0$, the right hand side admits a limit $\ell \in \mathbb{R} \cup \{+\infty\}$ as $z \to z^*$, and so does $(1 - u(z))u'(z)$. From (i), $\ell \in (-\infty, 0]$. Assume by contradiction $\ell < 0$ so that

$$((1 - u)^2)'(z) \sim -2\ell, \quad \text{as } z \to z^*. \tag{16}$$

If $z^* = -\infty$, it immediately follows, by integration of (16), that $(1 - u)^2(z)$ is unbounded as $z \to -\infty$, which is absurd. Let us now consider the case $-\infty < z^*$. In this case, by integration of (16), we reach, as $z \to z^*$, $(1 - u)(z) \sim \sqrt{-2\ell}\sqrt{z - z^*}$ which, in turn, provides $u'(z) \sim \frac{\ell}{\sqrt{-2\ell}} \frac{1}{\sqrt{z - z^*}}$. Plugging these two informations into the travelling wave equation (14), we then obtain, as $z \to z^*$, $u''(z) \sim \frac{c}{2(z - z^*)}$ (note that the case $c = 0$ is obviously excluded by the equation) which, after integration, provides $u'(z) \sim \frac{c}{2} \log(z - z^*)$, which contradicts a previous equivalent. Hence, in any case, $\ell = 0$ that is $(1 - u(z))u'(z) \to 0$ as $z \to z^*$. The limit $u'(+\infty) = 0$ may be obtained by similar arguments, or also via classical elliptic estimates.

As a result, we may let $z \to z^*$ and $t \to +\infty$ in (15) to reach

$$c = \int_{z^*}^{+\infty} (u'^2 + u(1 - u))(s)\, ds,$$

which concludes the proof of (ii) and (iii). $\square$

Thanks to the previous a priori facts we can show, in the following lemma, that we may indeed restrict our attention to the study of (14) in order to study travelling wave solutions to (1).

Lemma 3.2 (Equivalent formulations)**.** *Let (z^*, u) be a travelling wave profile in the sense of Definition 2.2. Let $c \in \mathbb{R}$ be given. Then the following are equivalent.*

(a) The function I defined by (6) *is a weak solution to* (1) *in the sense of Definition 2.1.*

(b) The function u satisfies (14).

Proof. Let us assume (a). In (5), we take $\varphi(t, x)$ as the tensor product of a test function $\psi \equiv \psi(x \cdot e - ct)$ with a test function depending on the directions orthogonal to the time-space vector $(-c, e_1, \ldots, e_N)$, one arrives at the following weak formulation of (14)

$$c\psi(z^*) + \int_{z^*}^{+\infty} \left[cu\psi' + (1 - u)u'\psi' - |u'|^2\psi - u(1 - u)\psi\right] \mathrm{d}z = 0\,, \tag{17}$$

which yields (14) after integrating by parts for ψ compactly supported in $(z^*, +\infty)$.

Let us assume (b). If u satisfies (14), then by Lemma 3.1, the corresponding I defined by (6) satisfies $I \in L^\infty(\mathbb{R} \times \mathbb{R}^N) \cap L^2_{\text{loc}}(\mathbb{R}, H^1_{\text{loc}}(\mathbb{R}^N))$. It remains to show that the weak formulation (5) is indeed satisfied. For this, one can plug the expression of I into it and check that it equivalently holds if for all smooth ψ supported in $\overline{(z^*, +\infty)}$, one has (17). Observe that unlike the previous implication here one has to allow test functions such that $\psi(z^*) \neq 0$ when z^* is finite. However this holds again by integration by parts using Lemma 3.1 for the boundary terms and the strong formulation (14). □

3.1 Reaching a first order ODE

We now aim at constructing travelling wave solutions. Our approach consists in transforming the problem into a first order ODE, a strategy already proved to be very relevant in related situations, see [33, 34], [30], [17] and the references therein. However, because of the degeneracy of the considered equation not in divergence form, the reached ODE is singular, in sharp contrast with [23] or [17], and infinitely many waves will co-exist.

From Lemma 3.1 (i), u is a C^2 bijection of $(z^*, +\infty)$ onto $(0, 1)$, and we may thus look for $u^{-1} : (0, 1) \to (z^*, +\infty)$. Obviously

$$(u^{-1})'(u(z)) = \frac{1}{u'(z)}. \tag{18}$$

Next, we define

$$h \coloneqq (u')^2 \circ u^{-1}\,. \tag{19}$$

By (18), we find

$$h' = 2u'' \circ u^{-1}$$

and replacing the expression for u'' derived from (14), we obtain

$$\frac{dh}{dr}(r) = -\frac{2c}{1-r} u' \circ u^{-1}(r) - 2r.$$

Thus we look for $h : (0, 1) \to (0, +\infty)$ solving

$$\begin{cases} \dfrac{dh}{dr} = \dfrac{2c}{1-r}\sqrt{h^+} - 2r \qquad 0 < r < 1, \\ h(0) = 0, \end{cases}$$

where $h^+ \coloneqq \max(h, 0)$ and where the initial datum $h(0) = 0$ is a direct consequence of the a priori fact $u'(+\infty) = 0$ proved in Lemma 3.1 (ii). Note also that the a priori fact $(1 - u(z))u'(z) \to 0$ as $z \to z^*$ proved in Lemma 3.1 (ii) enforces $(1 - r)^2 h(r) \to 0$ as $r \nearrow 1$ for any solution.

3.2 A Cauchy problem

In this subsection we thus investigate the Cauchy problem

$$\begin{cases} \dfrac{dh}{dr} = \dfrac{2c}{1-r}\sqrt{h^+} - 2r \quad 0 < r < 1, \\ h(0) = 0. \end{cases} \tag{20}$$

By Cauchy-Peano theorem, there exists at least one local solution, defined on some $[0, r^*)$ with $0 < r^* \leq 1$ (anyway, observe that $r \in [0, 1) \mapsto -r^2$ is a global solution!).

If a solution vanishes at some $r_0 > 0$ then, from the equation, $h'(r_0) = -2r_0 < 0$. As a result a solution can only be positive on $(0, r^*)$, or positive on some $(0, r_0)$-negative on (r_0, r^*), or negative on $(0, r^*)$ (and in this case this has to be $r \mapsto -r^2$).

We claim that any local solution is actually global. Indeed, $h' \geq -2r$ provides $h(r) \geq -r^2$, while $h' \leq \frac{2c}{1-r}\sqrt{h^+}$ provides $h(r) \leq c^2 (\log(1-r))^2$. Hence

$$-r^2 \leq h(r) \leq c^2 (\log(1-r))^2, \tag{21}$$

which prevents blow-up at $r^* < 1$. Hence $r^* = 1$ which proves the claim.

If a solution to (20) further satisfies $h > 0$ on $(0, 1)$, then this corresponds to a travelling wave through subsection 3.1.

We now discuss on the value of $c > 0$.

Lemma 3.3 (Small speeds). *If $0 < c < 2$ then the only solution of (20) is $r \mapsto -r^2$.*

Proof. Let $0 < c < 2$ and let us choose $\varepsilon > 0$ small enough that $c < \frac{2}{1+\varepsilon}$. Next select $r_\varepsilon > 0$ small enough so that

$$-\log(1-r) \leq (1+\varepsilon)r \quad \text{for } r \in (0, r_\varepsilon). \tag{22}$$

Assume by contradiction that there is a solution such that $h > 0$ on $(0, r_\varepsilon)$. It satisfies

$$\frac{d\sqrt{h}}{dr}(r) = \frac{c}{1-r} - \frac{r}{\sqrt{h(r)}}, \quad 0 < r < r_\varepsilon.$$

Integrating this and using (22), we obtain

$$\sqrt{h(r)} \leq c(1+\varepsilon)r - \int_0^r \frac{s}{\sqrt{h(s)}} \mathrm{d}s, \quad 0 < r < r_\varepsilon. \tag{23}$$

In particular $\sqrt{h(r)} \leq c(1+\varepsilon)r$, which can be plugged into (23) to obtain

$$\sqrt{h(r)} \leq \left(c(1+\varepsilon) - \frac{1}{c(1+\varepsilon)} \right) r.$$

By bootstrapping the bound one finds that

$$\sqrt{h(r)} \leq M_{n+1} r, \quad 0 < r < r_\varepsilon, \tag{24}$$

as long as M_n defined iteratively by

$$M_{n+1} = c(1+\varepsilon) - \frac{1}{M_n}, \qquad M_0 = c(1+\varepsilon), \tag{25}$$

is nonnegative. Observe that since $c < \frac{2}{1+\varepsilon}$, there is $\alpha < 1$ such that $c(1+\varepsilon) - x^{-1} \leq \alpha x$ for all $x > 0$. Therefore $M_n \leq M_0 \alpha^n$ and in particular there is a first $n_0 \in \mathbb{N}$ such that $M_{n_0} < \frac{1}{c(1+\varepsilon)}$. Then $M_{n_0+1} < 0$ and (24) is a contradiction. Hence, if $c < 2$, the only solution is negative in a neighborhood of 0 which leaves only $h : r \mapsto -r^2$. $\square$

We now work with $c \geq 2$. We define the relevant quantities

$$\lambda^- := \frac{c - \sqrt{c^2 - 4}}{2}, \quad \lambda^+ := \frac{c + \sqrt{c^2 - 4}}{2}, \tag{26}$$

solving $\lambda^2 - c\lambda + 1 = 0$, and note that $\frac{1}{c} < \lambda^- \leq \lambda^+ < c$, with equality if and only if $c = 2$.

As a preliminary result let us describe some sub- and supersolutions of (20). These results will be used repeatedly in the rest of the analysis. The proof of the lemma consists in straightforward computations and elementary inequalities and is omitted.

Lemma 3.4 (Sub- and supersolutions). *One has the following properties.*

(a) The function $r \mapsto c^2 (\log(1-r))^2$ is a strict supersolution of (20) on $(0,1)$.

(b) If $\lambda \in [\lambda^-, \lambda^+]$, then $r \mapsto \lambda^2 r^2 (1-r)^2$ is a strict subsolution of (20) on $(0,1)$.

(c) If $\lambda \in (0, \lambda^-) \cup (\lambda^+, +\infty)$, then $r \mapsto \lambda^2 r^2 (1-r)^2$ is a strict supersolution of (20) on $(0, r_\lambda)$ for some $r_\lambda > 0$.

(d) If $c \in (2, \frac{3\sqrt{2}}{2})$, $\alpha > 0$ and $\beta \in ((\lambda^-)^{-2} - 1, 1)$, then $r \mapsto (\lambda^-)^2 r^2 (1 + \alpha r^\beta)$ is a strict supersolution of (20) on $(0, r_{\alpha,\beta})$ for some $r_{\alpha,\beta} > 0$.

We start by constructing a "large" solution, in the sense that $h(1) = +\infty$, corresponding to a saturated wave.

Lemma 3.5 (A first "large" solution). *Assume $c \geq 2$. Then the Cauchy problem (20) has a solution H satisfying*

$$(\lambda^+)^2 r^2 (1-r)^2 < H(r) \leq c^2 (\log(1-r))^2, \quad 0 < r < 1. \tag{27}$$

Furthermore, this solution is increasing on $[0,1)$ and satisfies $H(1) = +\infty$, more precisely

$$H(r) \sim c^2 (\log(1-r))^2, \qquad \text{as } r \to 1. \tag{28}$$

Proof. It is straightforward to check that, for any $c > 0$, $\overline{h}(r) := c^2 (\log(1-r))^2$ is a strict supersolution on $(0,1)$ for the ODE, meaning $\overline{h}\,' > \frac{2c}{1-r}\sqrt{\overline{h}^{\,+}} - 2r$ on $(0,1)$. On the other hand, from Lemma 3.4, $\underline{h}(r) := (\lambda^+)^2 r^2 (1-r)^2$ is a strict subsolution on $(0,1)$ for the ODE meaning $\underline{h}\,' < \frac{2c}{1-r}\sqrt{\underline{h}^{\,+}} - 2r$ on $(0,1)$. One can check that $\underline{h} < \overline{h}$ on $(0,1)$, and the conclusion then follows from the classical monotone iteration method. To be more precise, we construct a sequence $(h_n)_{n\geq 0}$ of functions on $[0,1)$ by letting $h_0 \equiv \underline{h}$ and

$$h_{n+1}'(r) = \frac{2c}{1-r}\sqrt{h_n^+} - 2r, \quad h_{n+1}(0) = 0,$$

or, equivalently,

$$h_{n+1}(r) = 2c \int_0^r \frac{\sqrt{h_n^+(s)}}{1-s}\, ds - r^2, \quad 0 \leq r < 1. \tag{29}$$

Using that $\underline{h}$ and $\overline{h}$ are strict sub- and supersolutions, one can prove by induction that

$$\underline{h} \leq h_n \leq h_{n+1} \leq \overline{h}, \quad \text{for all } n \geq 0.$$

As a result, there is a function H on $[0,1)$ such that, for all $r \in [0,1)$, $h_n(r) \to H(r)$ as $n \to +\infty$. By the monotone convergence theorem, we can pass to the limit in the integral formulation (29) and reach the first conclusion.

Now, the constructed solution satisfies $H(r) \geq \underline{h}(r) = (\lambda^+)^2 r^2(1-r)^2 > \frac{r^2(1-r)^2}{c^2}$ so that, by the equation, $H'(r) > 0$. Hence $H(1)$ exists in $(0,+\infty]$. If $H(1) = \ell < +\infty$, then $H'(r) \sim \frac{2c\sqrt{\ell}}{1-r}$ as $r \to 1$ so that, by integration, $H(1) = +\infty$, a contradiction. Hence $H(1) = +\infty$ and $\frac{H'(r)}{2\sqrt{H(r)}} \sim \frac{c}{1-r}$ as $r \to 1$ so that, by integration, we obtain (28). □

Hence, for any $c \geq 2$, the solution H provided by Lemma 3.5 solves problem (20) (and, thus, already provides *one* saturated travelling wave, as detailed in subsection 3.4). Furthermore, by Cauchy-Lipschitz theorem, this H cannot be crossed by another solution and acts as a barrier.

In the sequel, a (r,h)-phase plane analysis provides valuable information. Observe first that, by the equation, the r-axis can only be crossed downhill. Next, the bell-shaped curve

$$B(r) := \frac{r^2(1-r)^2}{c^2}, \quad 0 \leq r \leq 1, \tag{30}$$

plays a central role. Indeed, a solution h decreases (resp. increases) when it is below (resp. above) the bell. Also, by using and differentiating the equation, we see that if a solution h touches the bell at some $0 < r < \frac{1}{2}$ (resp. $\frac{1}{2} < r < 1$) then it crosses it with $h'(r) = 0$, $h''(r) < 0$ (resp. $h'(r) = 0$, $h''(r) > 0$). Notice however that if a solution h touches the bell at $r = \frac{1}{2}$ then $h''(\frac{1}{2}) = 0$ and the bell is not crossed.

Equipped with the barrier H, we can now construct a "small" solution, in the sense that $h(1) = 0$, corresponding to the non-saturated wave.

Lemma 3.6 (A unique small solution)**.** *Assume $c \geq 2$. Then there is a unique solution h_0 to the Cauchy problem* (20) *that further satisfies $h_0(1) = 0$ (and it is positive on $(0,1)$). Also, there is $r_0 \in (0,\frac{1}{2})$ such that $h_0' > 0$ on $(0,r_0)$ while $h_0' < 0$ on $(r_0,1)$, and $h''(r_0) < 0$. Furthermore,*

$$\frac{r^2(1-r)^2}{c^2} < h_0(r) < (\lambda^-)^2 r^2(1-r)^2, \quad 0 < r < r_0, \tag{31}$$

and there is $\varepsilon > 0$ small enough such that

$$0 < h_0(r) < \frac{r^2(1-r)^2}{c^2}, \quad 1-\varepsilon < r < 1. \tag{32}$$

Proof. If there are two such solutions, h_1 and h_2, then the boundary conditions $(h_2-h_1)(0) = (h_2-h_1)(1) = 0$ enforce the existence of $r_0 \in (0,1)$ such that $(h_2-h_1)'(r_0) = 0$. From the equation this implies $h_1(r_0) = h_2(r_0)$ and then $h_1 \equiv h_2$ by Cauchy-Lipschitz theorem.

As for the existence, let us consider an increasing sequence $(r_n)_{n\geq 1}$ converging to 1. Denote h_n a solution to the Cauchy problem

$$\begin{cases} \dfrac{dh_n}{dr} = \dfrac{2c}{1-r}\sqrt{h_n^+} - 2r \quad 0 < r < 1, \\ h_n(r_n) = 0. \end{cases}$$

Since the r-axis can only be crossed downhill and $h_n'(r_n) = -2r_n < 0$, we have $0 < h_n(r) < H(r)$ for all $r \in (0,r_n)$, so that $h_n(0) = 0$. Also, on the right of r_n, there holds $h_n(r) = r_n^2 - r^2$

for all $(r_n, 1]$. Since two solutions can only cross on the $h = 0$ axis, we also have $h_n(r) < h_{n+1}(r) < H(r)$ for all $r \in (0, 1]$. As a result there is a function h_0 on $[0, 1]$ such that, for all $r \in [0, 1]$, $h_n(r) \to h_0(r)$ as $n \to +\infty$. In particular $h_0(0) = h_0(1) = 1$ and, as in the proof of Lemma 3.5, h_0 solves the ODE on $(0, 1)$.

From the phase plane analysis, h_0 has to be below the bell (30) in a neighborhood of 1, thus providing (32), has to cross the bell only once, at some $r_0 \in (0, \frac{1}{2})$, thus providing the sign of h_0'. As for (31), it follows again from the phase plane analysis and the fact that the right hand side in (31) is a strict subsolution that can only be crossed in one sense and that stands strictly above the bell (since $\lambda^- > \frac{1}{c}$). □

Hence, for any $c \geq 2$, the solution h_0 provided by Lemma 3.6 solves problem (20) (and, thus, already provides the non-saturated travelling wave, as detailed in subsection 3.4). Furthermore, this h_0 acts as a barrier for other possible solutions.

Equipped with h_0 and H, we can now construct infinitely many "large" solutions (providing infinitely many saturated waves). To do so, for a shooting parameter $\alpha > 0$, we consider the Cauchy problem

$$\begin{cases} \dfrac{dh}{dr} = \dfrac{2c}{1-r}\sqrt{h^+} - 2r \quad 0 < r < 1, \\ h(\frac{1}{2}) = \alpha, \end{cases} \tag{33}$$

whose solution is denoted h_α. If $\alpha < h_0(\frac{1}{2})$ then, from the above, h_α cannot solve problem (20) while being nonnegative. If $\alpha = h_0(\frac{1}{2})$ then $h_\alpha \equiv h_0$ the so-called small solution. If $\alpha = H(\frac{1}{2})$ then $h_\alpha \equiv H$ the so-called large solution constructed above. From using the solution to the ODE starting from any $h(0) > 0$ as a barrier, or the upper bound in (21), we see that

$$\alpha_{max} := \sup\{\alpha > 0 : h_\alpha(0) = 0\} < +\infty,$$

and that, using again monotony arguments as in the proof of Lemma 3.5, the supremum is reached, that is $h_{\alpha_{max}}(0) = 0$. Note also that

$$\alpha_{max} \geq H(1/2) > \frac{(\lambda^+)^2}{16}.$$

Lemma 3.7 (Infinitely many large solutions)**.** *Assume $c \geq 2$. Let*

$$h_0(1/2) < \alpha \leq \alpha_{max}. \tag{34}$$

Then the solution $h = h_\alpha$ to the Cauchy problem (33) *also solves the Cauchy problem* (20)*, $h > 0$ on $(0, 1)$,*

$$h(r) \sim c^2 (\log(1-r))^2 \qquad \text{as } r \to 1, \tag{35}$$

and

$$\frac{r^2(1-r)^2}{c^2} < h(r), \quad 0 < r < \varepsilon, \tag{36}$$

for some $\varepsilon > 0$. Furthermore, the following holds.

(a) If $h_0(\frac{1}{2}) < \alpha < \frac{1}{16c^2}$, then there are $0 < r_2 < \frac{1}{2} < r_1 < 1$ such that $h' > 0$ on $(0, r_2) \cup (r_1, 1)$ and $h' < 0$ on (r_2, r_1).

(b) If $\alpha = \frac{1}{16c^2}$, then $h' > 0$ on $(0, \frac{1}{2}) \cap (\frac{1}{2}, 1)$ and $h'(\frac{1}{2}) = 0$.

(c) If $\frac{1}{16c^2} < \alpha \leq \alpha_{max}$*, then* $h' > 0$ *on* $(0,1)$*.*

Proof. The proof is a direct consequence of the above considerations, including the phase plane analysis (note that $\frac{1}{16c^2} = B(1/2)$, the "top of the bell (30)"). $\square$

Some of the results that have been shown up to now are illustrated in Figure 2, where a few solutions are drawn together with the main reference curves that we used in our analysis.

3.3 Boundary behaviors of the h's

Hence, there is a one-to-one correspondence between travelling waves u (say normalized by $u(0) = \frac{1}{2}$) and the h_α's, with $h_0(1/2) \leq \alpha \leq \alpha_{max}$, provided by subsection 3.2.

Before returning to u, we precise the behavior of these h_α's as $r \to 0$ (which will translate later into an estimate on the tail of u as $z \to +\infty$). This requires a combination of the phase plane analysis, some sub- and supersolutions, and some bootstrapping arguments.

Lemma 3.8 (Not below λ^- as $r \to 0$)**.** *Assume* $c \geq 2$*. Take* $h = h_\alpha$ *a solution provided by Lemma 3.6 or Lemma 3.7. Then, for any* $0 < \varepsilon < \lambda^-$*, there is* $r_\varepsilon > 0$ *such that*

$$h(r) \geq (\lambda^- - \varepsilon)^2 r^2 (1-r)^2, \quad \forall r \in (0, r_\varepsilon).$$

Proof. Let $0 < \varepsilon \ll 1$ be fixed and let $r_0 \in (0,1)$ be such that the smallest root of $(1-r_0)^2 X^2 - cX + 1$ is larger than $\lambda^- - \varepsilon$. Up to taking a smaller r_0, by Lemma 3.4, $r \to (\lambda^- - \varepsilon)^2 r^2 (1-r)^2$ is also a strict supersolution on $(0, r_0)$. Now, assume by contradiction that there is $0 < r_1 < r_0$ such that

$$h(r) < \left(\lambda^- - \varepsilon\right)^2 r^2 (1-r)^2, \quad \forall r \in (0, r_1).$$

Using the integral formulation of the Cauchy problem, one can bootstrap and improve this bound to $0 < h(r) < K_n^2 r^2 (1-r)^2$, for all $n \in \mathbb{N}$, where the sequence $(K_n)_{n\in\mathbb{N}}$ is nonnegative, decreasing, and given by the induction

$$K_{n+1}^2 = \frac{cK_n - 1}{(1-r_0)^2}, \quad K_0 = \lambda^- - \varepsilon.$$

However such a sequence cannot converge since there is no root to $(1-r_0)^2 X^2 - cX + 1$ on the left of K_0, hence a contradiction. Therefore for any $0 < r_1 < r_0$, there is $r_\varepsilon \in (0, r_1)$ such that $(\lambda^- - \varepsilon)^2 r_\varepsilon^2 (1-r_\varepsilon)^2 \leq h(r_\varepsilon)$. As $r \to (\lambda^- - \varepsilon)^2 r^2 (1-r)^2$ is a strict supersolution on $(0, r_\varepsilon)$ (see Lemma 3.4), the lemma is proved. $\square$

Lemma 3.9 (Not above λ^+ as $r \to 0$)**.** *Assume* $c \geq 2$*. Take* $h = h_\alpha$ *a solution provided by Lemma 3.7. Then, for any* $\varepsilon > 0$*, there is* $r_\varepsilon > 0$ *such that*

$$h(r) \leq (\lambda^+ + \varepsilon)^2 r^2 (1-r)^2, \quad \forall r \in (0, r_\varepsilon).$$

Proof. First observe that there is $r_1 > 0$ such that (21) provides

$$h(r) \leq (2c)^2 r^2 (1-r)^2, \quad \forall r \in (0, r_1).$$

Second let $0 < \varepsilon \ll 1$ be fixed and let $r_0 \in (0,1)$ be such that the largest root of $(1 - r_0)^2 X^2 - cX + 1$ is smaller than $\lambda^+ + \varepsilon$. Let $r_\varepsilon = \min(r_0, r_1)$. A bootstrap argument as in the previous lemma improves the first bound inductively. Namely, for any $n \in \mathbb{N}$ there holds

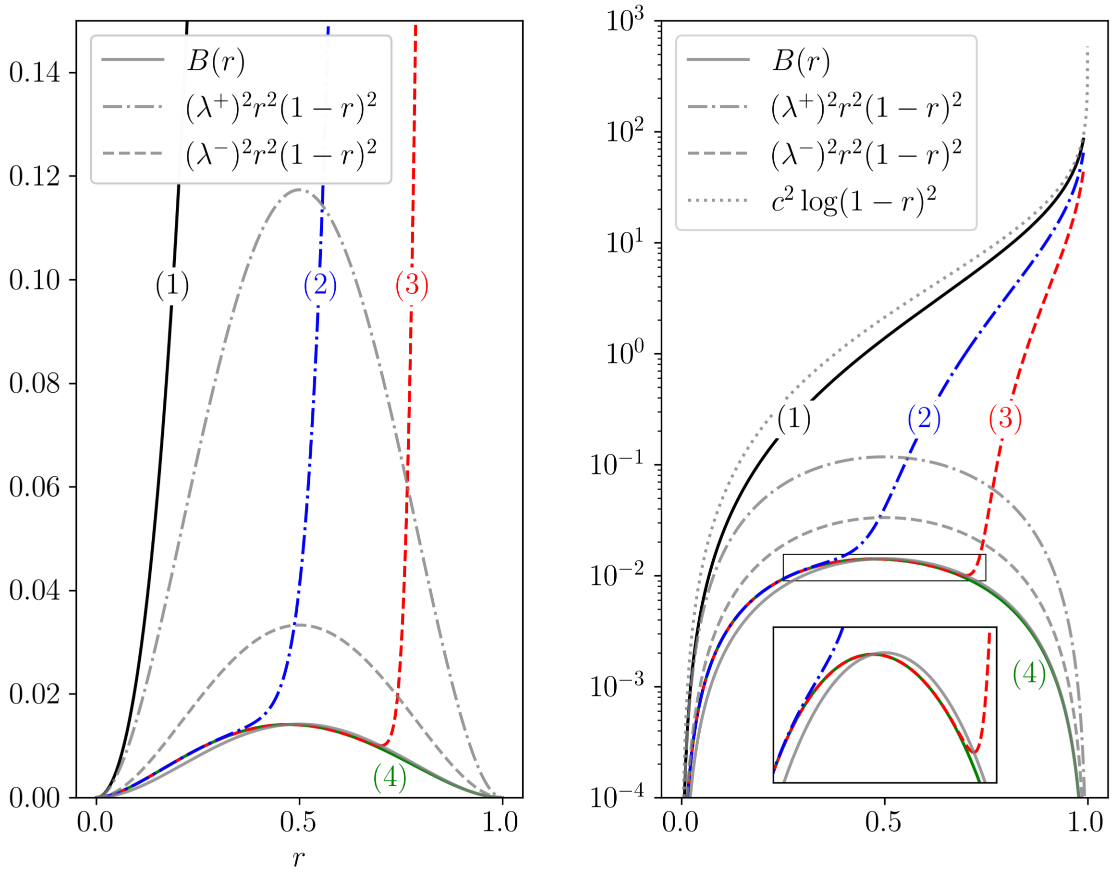

Figure 2: The (r, h)-phase plane in linear (left) and log-scale (right) for $c = 2.1$. The black, blue and red lines, labelled (1), (2) and (3), correspond to three different saturated wave solutions and are computed numerically. The green line, labelled (4), corresponds to the unique non-saturated wave. In particular, with respect to Lemma 3.7, (1) and (2) are two solutions of type (c), whereas (3) is a solution of type (a). With respect to Lemma 3.11, (1) is a solution of type (c), whereas (2) and (3) are solutions of type (a). These waves are represented in physical space in Figure 1.

$h(r) \leq K_n^2 r^2(1-r)^2$ for all $r \in (0, r_\varepsilon)$, where the sequence $(K_n)_{n\in\mathbb{N}}$ is positive, decreasing and given by the induction

$$K_{n+1}^2 = \frac{cK_n - 1}{(1-r_0)^2}\,, \quad K_0 = 2c.$$

Since K_n converges to a root of $(1-r_0)^2X^2 - cX + 1$ (which is smaller than $\lambda^+ + \varepsilon$), the lemma is proved. $\square$

We now consider the solutions h entering the region in-between the two bell-shaped subsolutions $r \mapsto \lambda^\pm r^2(1-r)^2$.

Lemma 3.10 (Sticking to λ^- as $r \to 0$)**.** *Assume $c \geq 2$. Take $h = h_\alpha$ a solution provided by Lemma 3.6 or Lemma 3.7. Assume*

$$\exists r_0 \in (0, 1/2],\ (\lambda^-)^2 r_0^2(1-r_0)^2 \leq h(r_0) \leq (\lambda^+)^2 r_0^2(1-r_0)^2. \tag{37}$$

Then, for any $0 < \varepsilon < \lambda^-$, there is $r_\varepsilon > 0$ such that

$$(\lambda^- - \varepsilon)^2 r^2(1-r)^2 \leq h(r) \leq (\lambda^- + \varepsilon)^2 r^2(1-r)^2, \quad \forall r \in (0, r_\varepsilon).$$

Proof. Let $0 < \varepsilon \ll 1$ be fixed. The lower bound comes from Lemma 3.8. Next, we recall that for any $\lambda \in [\lambda^-, \lambda^+]$, $r \mapsto \lambda^2 r^2(1-r)^2$ is a strict subsolution on $(0,1)$. Hence if $h(r_0) = (\lambda^-)^2 r_0^2(1-r_0)^2$ (which is necessarily the case for $c = 2$ due to (37), since in this case $\lambda^+ = \lambda^-$) then $h(r) \leq (\lambda^-)^2 r^2(1-r)^2$ on $(0, r_0)$ and we are already done. On the other hand, if $(\lambda^-)^2 r_0^2(1-r_0)^2 < h(r_0) \leq (\lambda^+)^2 r_0^2(1-r_0)^2$, then

$$\exists \lambda \in (\lambda^-, \lambda^+), \quad h(r) \leq \lambda^2 r^2(1-r)^2, \quad \forall r \in (0, r_0). \tag{38}$$

Assume (by contradiction) that there is $r_\varepsilon > 0$ such that

$$h(r) \geq (\varepsilon\lambda^+ + (1-\varepsilon)\lambda^-)^2 r^2(1-r)^2, \quad \forall r \in (0, r_\varepsilon).$$

Using this into the equation one finds, for $r \in (0, r_\varepsilon)$,

$$\begin{aligned} h(r) &= -r^2 + 2c\int_0^r \frac{\sqrt{h(s)}}{1-s}\mathrm{d}s \\ &\geq \big(c(\varepsilon\lambda^+ + (1-\varepsilon)\lambda^-) - 1\big)\, r^2 \\ &= \big(\varepsilon(\lambda^+)^2 + (1-\varepsilon)(\lambda^-)^2\big)\, r^2 \\ &\geq \big(\varepsilon(\lambda^+)^2 + (1-\varepsilon)(\lambda^-)^2\big)\, r^2(1-r)^2, \end{aligned}$$

a bound which is strictly better than the initial bound (by convexity). Iteratively one finds

$$h(r) \geq \big(\lambda^+\varepsilon_n + (1-\varepsilon_n)\lambda^-\big)^2 r^2(1-r)^2, \quad \forall r \in (0, r_\varepsilon),\ \forall n \in \mathbb{N},$$

with

$$\varepsilon_0 = \varepsilon, \quad \varepsilon_{n+1}(\lambda^+ - \lambda^-) + \lambda^- = \sqrt{\varepsilon_n(\lambda^+)^2 + (1-\varepsilon_n)(\lambda^-)^2}.$$

The sequence (ε_n) is increasing and, in view of (38), bounded by 1, so it converges to some $\varepsilon \leq \ell \leq 1$. Letting $n \to +\infty$ into the above recursive equation, we see that

$$(\ell\lambda^+ + (1-\ell)\lambda^-)^2 = \ell(\lambda^+)^2 + (1-\ell)(\lambda^-)^2,$$

which (recall $\lambda^- \neq \lambda^+$ since $c > 2$), by convexity, enforces $\ell = 1$, which contradicts (38).

As a result, there is a sequence $r_n \to 0$ such that

$$h(r_n) < \left(\lambda^- + \varepsilon(\lambda^+ - \lambda^-)\right)^2 r_n^2(1 - r_n)^2,$$

and the conclusion follows from the fact that $r \mapsto \left(\lambda^- + \varepsilon(\lambda^+ - \lambda^-)\right)^2 r^2(1 - r)^2$ is a strict subsolution on $(0, 1)$. $\square$

Combining the above, we get the picture described in the following lemma. In particular, note that in this lemma the large solution H is a solution of type (c) and the small solution h_0 is a solution of type (a).

Lemma 3.11 (Behavior of h_α as $r \to 0$)**.** *Assume $c \geq 2$. Take $h = h_\alpha$ a solution provided by Lemma 3.6 or Lemma 3.7. Then there exist*

$$\frac{(\lambda^+)^2}{16} < \alpha^-_{switch} \leq \alpha^+_{switch} \leq \alpha_{max},$$

with $\alpha^-_{switch} < \alpha^+_{switch}$ if $c \in (2, \frac{3\sqrt{2}}{2})$, such that

(a) If $h_0(\frac{1}{2}) \leq \alpha < \alpha^-_{switch}$ then the solution sticks to λ^- from below in the sense that, for any $0 < \varepsilon \ll 1$, there exists $r_\varepsilon > 0$ such that

$$(\lambda^- - \varepsilon)^2 r^2(1 - r)^2 \leq h(r) \leq (\lambda^-)^2 r^2(1 - r)^2, \quad \forall r \in (0, r_\varepsilon); \tag{39}$$

(b) If $\alpha^-_{switch} \leq \alpha < \alpha^+_{switch}$ then the solution sticks to λ^- from above in the sense that, for any $0 < \varepsilon \ll 1$, there exists $r_\varepsilon > 0$ such that

$$(\lambda^-)^2 r^2(1 - r)^2 < h(r) \leq (\lambda^- + \varepsilon)^2 r^2(1 - r)^2, \quad \forall r \in (0, r_\varepsilon); \tag{40}$$

(c) If $\alpha^+_{switch} \leq \alpha \leq \alpha_{max}$, then the solution sticks to λ^+ in the sense that, for any $0 < \varepsilon \ll 1$, there exists $r_\varepsilon > 0$ such that

$$(\lambda^+)^2 r^2(1 - r)^2 < h(r) \leq (\lambda^+ + \varepsilon)^2 r^2(1 - r)^2, \quad \forall r \in (0, r_\varepsilon).$$

Proof. Let us define

$$\alpha^\pm_{switch} := \sup \left\{ \alpha \geq \frac{(\lambda^\pm)^2}{16} : \exists r_0 \in (0, 1/2],\ h_\alpha(r_0) = (\lambda^\pm)^2 r_0^2(1 - r_0)^2 \right\}.$$

By continuity with respect to the parameter α, we have $\alpha^\pm_{switch} > (\lambda^\pm)^2/16$ and the supremum cannot be a maximum. The lower bound in point (a) follows from Lemma 3.8, and the upper bound from the fact that $r \mapsto (\lambda^-)^2 r^2(1 - r)^2$ is a strict subsolution on $(0, 1)$, as in the proof of Lemma 3.10. Point (b) is a direct application of Lemma 3.10 and the definition of α^-_{switch}. Let us show that this assumption of point (b) is not always empty, and more precisely that $\alpha^-_{switch} < \alpha^+_{switch}$ if $c \in (2, \frac{3\sqrt{2}}{2})$. For this, we consider $g^-(r) = (\lambda^- + \varepsilon)^2 r^2(1 - r)^2$ and $g^+(r) = (\lambda^-)^2 r^2(1 + \alpha r^\beta)$, for $\alpha > 0$ and $(\lambda^-)^{-2} - 1 < \beta < 1$ which are respectively sub- and supersolutions by Lemma 3.4. Moreover they are enclosed between the bells $r \to (\lambda^\pm)^2 r^2(1 - r)^2$ and satisfy that $g^- > g^+$, in some interval $(0, r_\varepsilon)$. Cauchy-Lipschitz theorem then allows to build infinitely many solutions $g^- > h > g^+$ which thus have the asymptotic behavior (40). These solutions correspond to h_α with $\alpha \geq \alpha^-_{switch}$ since $r \to (\lambda^-)^2 r^2(1 - r)^2$ is a subsolution, and $\alpha < \alpha^+_{switch}$ by definition of α^+_{switch}. Finally, the lower bound in point (c) also follows from the definition of α^+_{switch}, whereas the upper bound is a consequence of Lemma 3.9. $\square$

We now precise the behavior of h_0 as $r \to 1$ (which will translate later into an estimate on the tail of $1-u$ as $z \to -\infty$).

Lemma 3.12 (Sticking to c^{-1} as $r \to 1$)**.** *Assume $c \geq 2$. Take h_0 the solution provided by Lemma 3.6. Then for any $\varepsilon > 0$, there is $r_\varepsilon \in (0,1)$ such that*

$$\frac{r^2(1-r)^2}{(c+\varepsilon)^2} \leq h_0(r) < \frac{r^2(1-r)^2}{c^2}, \quad \forall r \in (r_\varepsilon, 1).$$

Proof. The upper bound has already been proved in Lemma 3.6. One can check that $r \mapsto \frac{r^2(1-r)^2}{(c+\varepsilon)^2}$ is a strict supersolution near $r=1$, say on some $(r_\varepsilon, 1)$, so that if $\frac{r^2(1-r)^2}{(c+\varepsilon)^2} > h_0(r)$ for some $r_\varepsilon^* \in (r_\varepsilon, 1)$ then the same inequality holds for *any* $r \in (r_\varepsilon^*, 1)$. Thus let us assume (by contradiction) that

$$h_0(r) \leq \frac{r^2(1-r)^2}{(c+\varepsilon)^2}, \quad \forall r \in (r_\varepsilon^*, 1).$$

Using this into the integral formulation

$$-h_0(r) = -(1-r^2) + 2c \int_r^1 \frac{\sqrt{h_0(s)}}{1-s} ds,$$

we obtain $h_0(r) \geq \frac{\varepsilon}{c+\varepsilon}(1-r^2) \geq \frac{\varepsilon}{c+\varepsilon}(1-r)$ which contradicts the upper bound in (32). □

3.4 Back to the wave profile u

For $c \geq 2$, let us now consider one of the infinitely many $h : (0,1) \to (0, +\infty)$ solving (20) constructed in subsection 3.2. Reversing the manipulations in subsection 3.1, we get a travelling wave. Precisely, it follows from (18) and (19) that

$$(u^{-1})'(r) = \frac{1}{u'(u^{-1}(r))} = -\frac{1}{\sqrt{h(r)}}.$$

We may normalize the travelling wave by $u(0) = \frac{1}{2}$ to, finally, reach

$$w(r) := u^{-1}(r) = \int_r^{\frac{1}{2}} \frac{ds}{\sqrt{h(s)}}, \quad 0 < r < 1. \tag{41}$$

Furthermore, the quadratic behavior of h at $r \to 0$ corresponds to the exponential decay of the wave towards 0 as $z \to +\infty$. Indeed, assume that, for some $0 < \varepsilon \leq \frac{1}{2}$ and $0 < a < b$, we have

$$a^2 r^2 (1-r)^2 \leq h(r) \leq b^2 r^2 (1-r)^2, \quad 0 < r < \varepsilon.$$

Then it follows from (41) that

$$\frac{1}{b}\left(\log \frac{\varepsilon}{1-\varepsilon} - \log \frac{r}{1-r}\right) \leq w(r) - C \leq \frac{1}{a}\left(\log \frac{\varepsilon}{1-\varepsilon} - \log \frac{r}{1-r}\right), \quad 0 < r < \varepsilon,$$

where $C := \int_\varepsilon^{1/2} \frac{ds}{\sqrt{h(s)}} > 0$. Returning to u via $r = u(z)$ this is transferred to

$$C_1 e^{-bz} \leq u(z) \leq C_2 e^{-az}, \quad z > z_0,$$

for some large enough $z_0 > 0$, and some constants $C_1, C_2 > 0$.

As for the behavior "on the left", there are two possibilities. First, if

$$\int_{\frac{1}{2}}^{1} \frac{ds}{\sqrt{h(s)}} = +\infty, \tag{42}$$

this corresponds to a non-saturated wave (i.e. $z^* = -\infty$ in the setting of Definition 2.2). In view of (32), this occurs for the small solution h_0 of Lemma 3.6. Similarly as above, the quadratic behavior of h at $r \approx 1$ informs on the exponential decay of the wave towards 1 as $z \to -\infty$: one can check that

$$a^2 r^2 (1-r)^2 \leq h(r) \leq b^2 r^2 (1-r)^2, \quad 1 - \varepsilon < r < 1,$$

is transferred to

$$C_1 e^{bz} \leq 1 - u(z) \leq C_2 e^{az}, \quad z < -z_0,$$

for some large enough $z_0 > 0$, and some constants $C_1, C_2 > 0$.

On the other hand, if (42) does not hold, this corresponds to a saturated wave reaching 1 at

$$-\infty < z^* = -\int_{\frac{1}{2}}^{1} \frac{ds}{\sqrt{h(s)}} < 0.$$

In view of (35), this occurs for the infinitely many large solutions of Lemma 3.6. Furthermore, since these h's satisfy (35), we have

$$u'(z) = \frac{1}{w'(u(z))} = -\sqrt{h(u(z))} \sim c \log(1 - u(z)), \quad \text{as } z \searrow z^*,$$

which, by integration, provides

$$-\mathrm{li}(1 - u(z)) \sim c(z - z^*), \quad \text{as } z \searrow z^*,$$

where $\mathrm{li}(x) \coloneqq \int_0^x \frac{ds}{\log s}$ is the logarithmic integral function. Since $\mathrm{li}(x) \sim \frac{x}{\log x}$ as $x \to 0$, this is transferred to

$$\frac{1 - u_S(z)}{-\log(1 - u_S(z))} \sim c(z - z^*), \quad \text{as } z \searrow z^*. \tag{43}$$

Writing $1 - u_S(z) = \frac{c(z - z^*)}{\psi(z)}$ with $\psi(z) \to 0$ as $z \to z^*$, so that $\log \psi(z) = \log(c(z - z^*)) - \log(1 - u_S(z))$, and then using (43), one reaches $\frac{1}{\psi(z)} \sim -\log(z - z^*)$, and thus (9). Note in particular that $u'(z^*) = -\infty$ and the wave is sharp.

One may also check that, for all $z \in (z^*, +\infty)$,

$$u''(z) = \frac{1}{2} h'(u(z)),$$

so that a zero for h' corresponds to a zero for u''. Furthermore, if $z \in (z^*, +\infty)$ is such that $u''(z) = 0$ then $u'''(z) = \frac{1}{2} u'(z) h''(u(z))$, which is not zero if we further assume $u(z) \neq \frac{1}{2}$ (meaning that h is not the solution tangent at the top of the bell). In other words, if $u(z) \neq \frac{1}{2}$ and $u''(z) = 0$, then z is an inflection point.

Conclusion. Based on these observations, it is straightforward to check that the many $h : (0,1) \to (0,+\infty)$ solving (20) constructed in subsection 3.2 provide the infinitely many travelling waves as stated in Theorem 2.4. To be more precise, u_{NS} in Theorem 2.4 (i) comes from the small solution of Lemma 3.6, while the u_S's in Theorem 2.4 (ii) come from the large solutions of Lemma 3.7. As for the estimates of the convergence rate to 0 or 1, they are provided by the analysis in subsection 3.3. $\square$

Remark 3.13. *Let us finally precise that the estimate* (11) *in Theorem 2.4 can be made slightly more precise, based on the conclusions of Lemma 3.11. Indeed, among the strictly convex saturated travelling wave profiles satisfying* (11)*, one has that infinitely many satisfy*

$$C_{11}\, e^{-\lambda^- z} \leq u_S(z) \leq C_{12}\, e^{-(\lambda^- - \varepsilon)z}, \qquad \text{as } z \to +\infty.$$

For small admissible velocities $c \in (2, \frac{2\sqrt{3}}{2})$, there are also infinitely many satisfying

$$C_{13}\, e^{-(\lambda^- + \varepsilon)z} \leq u_S(z) \leq C_{14}\, e^{-\lambda^- z}, \qquad \text{as } z \to +\infty.$$

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